THE RAMSEY-TURÁN PROBLEM FOR CLIQUES

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Dedicated to Vera T. Sós

Abstract. An important question in extremal graph theory raised by Vera T. Sós asks to determine for a given integer $t \geqslant 3$ and a given positive real number δ the asymptotically supremal edge density $f_t(\delta)$ that an n-vertex graph can have provided it contains neither a complete graph K_t nor an independent set of size δn.

Building upon recent work of Fox, Loh, and Zhao [*The critical window for the classical Ramsey-Turán problem*, Combinatorica **35** (2015), 435–476], we prove that if δ is sufficiently small (in a sense depending on t), then

$$f_t(\delta) = \begin{cases} \frac{3t-10}{3t-4} + \delta - \delta^2 & \text{if } t \text{ is even,} \\ \frac{t-3}{t-1} + \delta & \text{if } t \text{ is odd.} \end{cases}$$

§1. Introduction

P. Turán [15] established a new subarea of extremal combinatorics nowadays bearing his name. In the context of graphs, the fundamental question he proposed is to determine, for a given positive number n and a given graph F, the maximum number $\mathrm{ex}(n, F)$ of edges that a graph of order n can have provided that it does not contain F as a subgraph. Turán himself gave the complete answer if F is a clique, and an asymptotically satisfactory solution for all graphs F has been obtained by the work of Erdős, Stone, and Simonovits (see [4, 6]). Curiously, the corresponding problem for hypergraphs is wide open, even in the 3-uniform case.

Another branch of combinatorics related to our discussion, called Ramsey theory, was initiated by F. P. Ramsey [11] and since then it has been developed into a coherent and successful body of results. A somewhat special yet typical case of Ramsey's original theorem asserts that if n is large enough depending on k, then no matter how one colours the edges of a complete graph of order n using two colours, there will always be a monochromatic complete subgraph of order k.

Vera T. Sós discovered a beautiful way of combining Ramsey theory with Turán theory by asking and investigating the following question: Given a positive integer n, a positive

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real number m, and a graph F, what is the maximum number $\mathrm{RT}(n, m, F)$ of edges that a graph G of order n can have if it does not contain F as a subgraph and $\alpha(G) < m$, i.e., if any $X \subseteq V(G)$ with $|X| \geqslant m$ spans at least one edge?

For example, if $m = n$ and F has at least one edge, then the condition on independent sets becomes vacuous and one recovers Turán's original problem, i.e., one has $\mathrm{RT}(n, n, F) = \mathrm{ex}(n, F)$. On the other hand, if m is very small, then by Ramsey's theorem each graph of order n contains either a clique of order $v(F)$ (and hence, in particular, a subgraph isomorphic to F) or an independent set of order $\lceil m \rceil$, meaning that the definition of $\mathrm{RT}(n, m, F)$ degenerates to the "maximum of the empty set." Using a quantitative version of Ramsey's theorem, this can be seen to happen, e.g., if $m < n^{1/v(F)}$ and n is large. So for fixed n and F the problem of determining $\mathrm{RT}(n, m, F)$ is mostly dominated by Ramsey theoretic phenomena for very small m and by Turán theory for very large m. If m is of medium size, however, the problem intriguingly combines the flavours of both areas. For further information on Ramsey-Turán theory the reader is referred to the comprehensive survey [12] by Simonovits and Sós.

In this article we restrict our attention to the perhaps most classical case that $m = \delta n$ for some small $\delta > 0$ and $F = K_t$ is a clique. To eliminate minor fluctuations arising from small values of n one usually focuses on the *Ramsey-Turán density function* $f_t \colon (0,1) \longrightarrow \mathbb{R}$ defined by

$$f_t(\delta) = \lim_{n\to\infty} \frac{\mathrm{RT}(n, \delta n, K_t)}{n^2/2}\,.$$

It is well known and easy to confirm that this limit does indeed exist. Since f_t is evidently a nondecreasing function of δ, a further simplification may be achieved by passing to the *Ramsey-Turán density* $\varrho(K_t)$ defined by

$$\varrho(K_t) = \lim_{\delta\to 0} f_t(\delta)\,.$$

Perhaps surprisingly at first, the difficulty of determining the quantities just introduced depends significantly on the parity of t. The first case where something happens is $t = 3$. One has $\mathrm{RT}(n, \delta n, K_3) \leqslant \delta n^2/2$ because if a graph G of order n has a vertex x whose degree is at least δn, then either the neighbourhood of x is independent, which gives $\alpha(G) \geqslant \delta n$, or this neighbourhood spans an edge yz, in which case xyz is a triangle. This simple observation implies $f_3(\delta) \leqslant \delta$ for all $\delta > 0$. Explicit examples described by Brandt [2] show that for $\delta < \frac{1}{3}$ this bound is optimal (see Proposition 2.1 and also Corollary 2.2 below), i.e., that we have $f_3(\delta) = \delta$ for all $\delta \in \big(0, \frac{1}{3}\big)$; in particular, $\varrho(K_3) = 0$. Concerning larger odd cliques, Erdős and Sós [5] proved $\varrho(K_{2r+1}) = \frac{r-1}{r}$ for all positive integers r, and a quantitative version of their argument yields

$$\frac{r-1}{r} \leqslant f_{2r+1}(\delta) \leqslant \frac{r-1}{r} + 2\delta$$

for all positive δ.

The first result addressing an even clique was obtained by Szemerédi [13], who proved that $\varrho(K_4) \leqslant \frac{1}{4}$. At that moment it still seemed conceivable that the truth might be $\varrho(K_4) = 0$. But a few years later Bollobás and Erdős [1] ruled out this possibility by exhibiting a remarkable geometric construction demonstrating the optimality of Szemerédi's bound; that is they completed the proof of $\varrho(K_4) = \frac{1}{4}$. Still later the Ramsey-Turán densities of all even cliques were determined by Erdős, Hajnal, Sós, and Szemerédi [3], the answer being

$$\varrho(K_{2r}) = \tfrac{3r-5}{3r-2} \quad \text{for all } r \geqslant 2\,. \tag{1.1}$$

The understanding as to how fast $f_4(\delta)$ converges to $\frac{1}{4}$ developed as follows. Szemerédi's original argument yields

$$f_4(\delta) \leqslant \tfrac{1}{4} + O\left(\left(\log\log \tfrac{1}{\delta}\right)^{-1/2+o(1)}\right)\,.$$

Conlon and Schacht observed independently in unpublished work that the Frieze-Kannan regularity lemma from [7] can be used to improve this to

$$f_4(\delta) \leqslant \tfrac{1}{4} + O\left(\left(\log \tfrac{1}{\delta}\right)^{-1/2}\right)\,.$$

Significant further progress is due to Fox, Loh, and Zhao [8], who obtained

$$\tfrac{1}{4} + \delta - \delta^2 \leqslant f_4(\delta) \leqslant \tfrac{1}{4} + 3\delta \tag{1.2}$$

for sufficiently small δ and asked

(*1*) how this gap can be narrowed down further
(*2*) and whether comparable results could be proved for larger even cliques and, in particular, whether $f_{2r}(\delta) = \varrho(K_{2r}) + \Theta(\delta)$ holds for all $r \geqslant 2$.

Our main result addresses both questions. Much to our own surprise, it turned out that at least for $\delta \ll r^{-1}$ there is a precise formula for the values of the Ramsey-Turán density function.

Theorem 1.1. *If* $r \geqslant 2$ *and* $\delta \ll r^{-1}$, *then* $f_{2r}(\delta) = \frac{3r-5}{3r-2} + \delta - \delta^2$.

The hard part of this result is the upper bound and we would like to restate it here in an elementary form, i.e., without talking about the function f_{2r}.

Theorem 1.2. *For every integer* $r \geqslant 2$ *there exists a real number* $\delta_* > 0$ *such that if* $\delta \leqslant \delta_*$, *then every graph* G *on* n *vertices with*

$$\alpha(G) < \delta n \quad \text{and} \quad e(G) > \left(\tfrac{3r-5}{3r-2} + \delta - \delta^2\right)\tfrac{n^2}{2}$$

contains a K_{2r}.

Incidentally, such an exact formula does also hold for odd cliques.

Theorem 1.3. *If* $r \geqslant 1$ *and* $\delta \ll r^{-1}$, *then* $f_{2r+1}(\delta) = \frac{r-1}{r} + \delta$.

Organisation. The lower bound constructions establishing that $f_t(\delta)$ has at least the value claimed in Theorem 1.1 and Theorem 1.3 are given in Section 2. The proof of Theorem 1.2 constitutes the main part of this article and occupies the Sections 3–6. A brief outline of its main steps becomes more informative as soon as one knows the corresponding extremal construction, and thus we defer such an overview to the end of Section 2. We conclude by giving in Section 7 a brief sketch of the proof of the upper bound in Theorem 1.3.

§2. THE LOWER BOUND

The goal of this section is to verify the lower bounds on $f_t(\delta)$ from Theorem 1.1 and Theorem 1.3 by means of explicit constructions. To this end, we just need to combine some results from [2] and [8].

We begin by recapitulating [2, Theorem 2.1]. This statement deals with the set Ω of all pairs (d, n) of natural numbers for which there exists a triangle-free, d-regular graph on n vertices with independence number d. Of course, if $(d, n) \in \Omega$, then $\mathrm{RT}(n, d+1, K_3) = \frac{1}{2}dn$ is as large as possible.

A standard blow-up argument shows that if $(d, n) \in \Omega$, then all multiples of this pair belong to Ω as well, that is we have $(ad, an) \in \Omega$ for all $a \in \mathbb{N}$. This suggest that rather than studying Ω itself one may want to focus on the set of quotients

$$S = \left\{\tfrac{d}{n} \colon (d, n) \in \Omega\right\}.$$

Brandt [2] discovered constructions which show the following.

Proposition 2.1. *The set* $S \cap \big(0, \frac{1}{3}\big)$ *is dense in* $\big(0, \frac{1}{3}\big)$. *Moreover,* $\big(0, \frac{7}{30}\big) \cap \mathbb{Q}$ *and* $\big(\frac{1}{4}, \frac{1}{3}\big) \cap \mathbb{Q}$ *are subsets of* S.

The "moreover"-part is not going to be used in the sequel and has been included here for the readers information only.

Corollary 2.2. *For fixed* $r \geqslant 1$ *and* $\delta < \frac{1}{3r}$ *we have*

$$\mathrm{RT}(n, \delta n, K_{2r+1}) \geqslant \big(\tfrac{r-1}{r} + \delta - o(1)\big)\tfrac{n^2}{2}\,.$$

Proof. Let $\eta > 0$ be given. We need to show that $\mathrm{RT}(n, \delta n, K_{2r+1}) \geqslant \big(\frac{r-1}{r} + \delta - \eta\big)\frac{n^2}{2}$ holds for all sufficiently large integers n. By Proposition 2.1 there exists a pair $(d_*, n_*) \in \Omega$ such that $\frac{d_*}{n_*} \in \big(r(\delta - \eta), r\delta\big)$. Now it suffices to show that

$$\mathrm{RT}(arn_*, ad_* + 1, K_{2r+1}) \geqslant \left(\frac{r-1}{r} + \frac{d_*}{rn_*}\right)\frac{(arn_*)^2}{2} \tag{2.1}$$

holds for every $a \in \mathbb{N}$. This is because for sufficiently large n we can add at most rn_* isolated vertices to a graph establishing (2.1), thus obtaining the desired lower bound on $\mathrm{RT}(n, \delta n, K_{2r+1})$.

To prove (2.1) we use $(ad_*, an_*) \in \Omega$ and take a triangle-free, (ad_*)-regular graph H on an_* vertices with $\alpha(H) = ad_*$. Now let $V = V_1 \cup \ldots \cup V_r$ be a disjoint union of r vertex classes each of which has size an_*, and construct a graph G on V

- inducing on each vertex class V_i a graph isomorphic to H,
- in which any two vertices from different classes are adjacent.

From $K_3 \nsubseteq H$ and the box principle it follows that $K_{2r+1} \nsubseteq G$. Every subset of V which is independent in G needs to be contained in a single vertex class, whence

$$\alpha(G) = \alpha(H) < ad_* + 1 .$$

Finally, we have

$$e(G) = \binom{r}{2}(an_*)^2 + re(H) = \left(\frac{r-1}{r} + \frac{d_*}{rn_*}\right)\frac{(arn_*)^2}{2} .$$

Therefore, G has all the properties necessary for witnessing (2.1). $\square$

Let us proceed with essentially extremal examples for even cliques. As mentioned in the introduction, Bollobás and Erdős [1] found a geometric construction showing that $\mathrm{RT}(n, o(n), K_4) \geqslant \left(\frac{1}{4} + o(1)\right)\frac{n^2}{2}$. The vertex set of their graph splits into two subsets of size $\frac{n}{2}$ inducing triangle-free graphs with $o(n^2)$ edges. Between those sets, called A and B from now on, there is a very special quasirandom bipartite graph of density $\frac{1}{2} - o(1)$.

To aid the readers orientation we remark that the graphs induced by A and B are not only triangle-free. As a matter of fact, they are “locally bipartite” in the sense of having rather large odd-girth. In particular, they do not contain cycles of length 5 or 7. Such properties will also play an important rôle in our proof of the upper bound (see Fact 5.7.2 below).

It is not entirely straightforward to make the asymptotic expressions in the result of Bollobás and Erdős explicit. The best quantitative analysis we are aware of has been conducted by Fox, Loh, and Zhao [8, Corollary 8.9], who obtained the following.

Theorem 2.3. *If n is sufficiently large and $\xi = 4(\log\log n)^{3/2}/(\log n)^{1/2}$, then*

$$\mathrm{RT}(n, \xi n, K_4) \geqslant \left(\tfrac{1}{8} - \xi\right)n^2 .$$

Let us proceed with a discussion of [8, Theorem 1.7] and the remark thereafter. Suppose that $\delta \in \left(0, \frac{1}{2}\right)$ is fixed and that n is a sufficiently large and (just for transparency) even natural number. Let G be a graph on n vertices as obtained by Theorem 2.3. Recall that there is a partition $V(G) = A \cup B$ with $|A| = |B| = \frac{n}{2}$ of its vertex set into two subsets not inducing triangles. Let $X \subseteq A$ and $Y \subseteq B$ be two random sets of size $|X| = |Y| = (\delta - \xi)n$, and let G_* be the graph obtained from G by removing all edges incident with $X \cup Y$ and then adding all edges from X to B as well as all edges from Y to A. Surely, G_* is K_4-free and all its independent sets have size less than δn. Moreover, a short calculation displayed in the proof of [8, Lemma 9.1] shows that the expected number of edges of G_* is at least $\big(\frac{1}{4} + \delta - \delta^2 - o(1)\big)\frac{n^2}{2}$. Therefore, we have indeed $f_4(\delta) \geqslant \frac{1}{4} + \delta - \delta^2$.

This construction combines with [3, Theorem 5.4] in the following way.

Proposition 2.4. *If* $r \geqslant 2$ *and* $\delta \in \big(0, \frac{1}{3r-2}\big)$ *are fixed, then*

$$\mathrm{RT}(n, \delta n, K_{2r}) \geqslant \big(\tfrac{3r-5}{3r-2} + \delta - \delta^2 - o(1)\big)\tfrac{n^2}{2}\,.$$

Proof. Let n be sufficiently large and, without loss of generality, divisible by $3r-2$. Take a set V of n vertices as well as a partition

$$V = V_1 \cup V_2 \cup \ldots \cup V_r \tag{2.2}$$

with $|V_i| = \frac{2}{3r-2}n$ for $i = 1, 2$ and $|V_i| = \frac{3}{3r-2}n$ for $i = 3, \ldots, r$. Construct a graph G on V whose edges are as follows.

- The subgraph of G induced by $V_1 \cup V_2$ is the graph described above exemplifying the lower bound
$$\mathrm{RT}\big(\tfrac{4}{3r-2}n, \delta n, K_4\big) \geqslant \tfrac{2}{(3r-2)^2}n^2 + \tfrac{2}{3r-2}\delta n^2 - \tfrac{1}{2}\delta^2 n^2 - o(n^2)\,,$$
the sets V_1 and V_2 here playing the rôles of A and B there.
- For $i \in [3, r]$ the graph that G induces on V_i is obtained by Corollary 2.2 and demonstrates
$$\mathrm{RT}\big(\tfrac{3}{3r-2}n, \delta n, K_3\big) \geqslant \tfrac{3}{2(3r-2)}\delta n^2 - o(n^2)\,.$$
- If $1 \leqslant i < j \leqslant r$ and $(i, j) \neq (1, 2)$, then all pairs uv with $u \in V_i$ and $v \in V_j$ are edges of G.

Evidently, every clique in G can have at most three vertices in $V_1 \cup V_2$ and at most two vertices in each V_i with $i \in [3, r]$, which proves that G is K_{2r}-free. Moreover, each independent subset of V is either contained in $V_1 \cup V_2$ or in one of the sets V_i with $i \in [3, r]$. Consequently, we have $\alpha(G) < \delta n$. Finally, a quick computation shows

$$\begin{aligned} 2e(G) &= \Big[\big(\tfrac{4}{(3r-2)^2} + \tfrac{4}{3r-2}\delta - \delta^2\big) + \tfrac{3(r-2)}{3r-2}\delta + \tfrac{9(r-2)(r-3)+24(r-2)}{(3r-2)^2} - o(1)\Big]n^2 \\ &= \big(\tfrac{3r-5}{3r-2} + \delta - \delta^2 - o(1)\big)n^2\,. \end{aligned}$$

So altogether G has all required properties. □

We conclude this section with a brief discussion of the strategy we use for proving Theorem 1.2 in the next four sections.

The first observation is that the process of repeatedly removing vertices of small degree allows us to focus on the case that $\delta(G) \geqslant \frac{3r-5}{3r-2}n$. Besides, by making further sacrifices as to the eventual value of δ_*, we can always assume that n is sufficiently large. For these reasons, it suffices to prove Proposition 6.1 below and large parts of Section 6 deal with the implication from Proposition 6.1 to Theorem 1.2.

So let us suppose we have a sufficiently large K_{2r}-free graph G with $\delta(G) \geqslant \frac{3r-5}{3r-2}n$ and $\alpha(G) < \delta n$, where δ is extremely small. Our task is to prove the upper bound $e(G) \leqslant \bigl(\frac{3r-5}{3r-2} + \delta - \delta^2\bigr)\frac{n^2}{2}$ on the number of its edges.

The argument starts similar to the proof of (1.1) given in [3]. That is we apply Szemerédi's regularity lemma and try to find one of several configurations in the regular partition, each of which would allow us to embed a K_{2r}. In [3] this is done by applying some Turán theoretic result to the reduced graph (see [3, Lemma 3.3]) and the assumed absence of these configurations leads to an upper bound of the form $e(G) \leqslant \bigl(\frac{3r-5}{3r-2} + \delta'\bigr)\frac{n^2}{2}$ with $\delta' \to 0$ as $\delta \to 0$.

However, since for a given δ we are aiming at a somewhat better estimate on $e(G)$ than [3] does, it may happen to us that this argument does not lead to immediate success. Yet there is still something we can do in order to proceed. Namely, we can prove a stability version of [3, Lemma 3.3], apply it to the reduced graph, and transfer the information thus obtained back to the original graph. In this manner, it can be shown that, in an approximate sense, our graph G does almost look like the extremal graph described in the proof of Proposition 2.4. Specifically, we find a partition

$$V(G) = A_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} A_r \tag{2.3}$$

such that each partition class spans at most $o(n^2)$ edges and the edge density between A_1 and A_2 is, in a hereditary sense, at most $\frac{1}{2} + o(1)$ (see Proposition 3.1 below for a precise statement). Utilising the lower bound $e(G) \geqslant \frac{3r-5}{3r-2} \cdot \frac{n^2}{2}$, which follows from the minimum degree condition, one can prove that these two conditions imply that the partition classes $A_1, \ldots, A_r$ have roughly the expected sizes and that, as long as $\{i, j\} \neq \{1, 2\}$, almost all possible edges between A_i and A_j are present in G (see Fact 4.2 below).

When one applies Proposition 3.1 to the essentially extremal graph constructed above, one ends up getting a partition which is to some extent similar to (2.2), but it does not necessarily agree with it. More precisely, one could show that, perhaps after an appropriate permutation of the indices, one has $\sum_{i=1}^{r} |A_i \triangle V_i| = o(n)$. But the constant implied in the o-notation here could be extremely large in comparison to δ and thus it seems desirable to produce a better partition before one starts deriving the asymptotically optimal upper bound on $e(G)$.

Constructing such an improved partition is the subject of Subsection 4.2. Its main result, Proposition 4.4, tells us that the graph G under consideration admits a so-called *exact partition* $V(G) = B_1 \charfusion[\mathbin]{\cup}{\cdot} \dots \charfusion[\mathbin]{\cup}{\cdot} B_r$ satisfying a long list of properties enumerated in Definition 4.3. These conditions are rather restrictive and it might be helpful to imagine that, up to a relabeling of the indices, (2.2) is the only exact partition of the extremal graph. The proof of Proposition 4.4 starts from the partition (2.3) and is based on an iterative procedure that moves vertices around that do not properly fit into the partition class they currently belong to.

Finally, in Section 5 we address the question how the knowledge of an exact partition allows us to prove an upper bound on $e(G)$ (see Proposition 5.2). The starting point there is the equation

$$2e(G) = \sum_{i=1}^{r} e(B_i, V)\,.$$

It turns out that one can separately show upper bounds for each of these terms, namely

$$e(B_i, V) \leqslant |B_i|(n - |B_1| - |B_2|) + \tfrac{1}{2}|B_1||B_2| + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2 \tag{2.4}$$

for $i = 1, 2$ (see Claim 5.7 below) and

$$e(B_i, V) \leqslant |B_i|(n - |B_i|) + \delta n|B_i| \tag{2.5}$$

for $i = 3, \dots, r$ (see Claim 5.5). By adding these estimates and optimising over $\sum_{i=1}^{r} |B_i| = n$ one obtains the desired bound $e(G) \leqslant \bigl(\frac{3r-5}{3r-2} + \delta - \delta^2\bigr)\frac{n^2}{2}$.

Notice that there are two cases in which (2.5) is rather easy. First, if B_i happens to be triangle-free, we get $e(B_i) \leqslant \frac{1}{2}\delta n|B_i|$ from $\alpha(G) < \delta n$ and by adding the trivial upper bound $e(B_i, V \smallsetminus B_i) \leqslant |B_i|(n - |B_i|)$ the claim follows. Second, if it happens that B_i misses at least $2\varepsilon n^2$ edges to $V \smallsetminus B_i$ for an appropriate (absolute) constant $\varepsilon > 0$, then the weaker bound $e(B_i) \leqslant \varepsilon n^2$, which exact partitions always satisfy, is enough to deduce (2.5). The general argument is a superposition of these two cases. That is, we will define a partition of B_i into a triangle-free part B_i^+ to which the first argument applies and another part B_i^- that misses sufficiently many edges to $V \smallsetminus B_i$ to make the second approach useful.

The estimate (2.4) is much harder. Let us focus here on the case $r = 2$ and $i = 1$, in which many of the difficulties are already visible. To keep this overview simple we will also assume that every vertex in B_1 sends at least $\frac{1}{2}|B_2| - \frac{1}{60}n$ edges to B_2. Recall that in the extremal example there is a set $S \subseteq B_1$ of size close to δn whose members are complete to B_2, whilst each vertex in $B_1 \smallsetminus S$ sends a little bit less than $\frac{1}{2}(|B_2| + \delta n)$ edges to B_2. Moreover, there is only a negligible number of edges within B_1. To prove (2.4), we can *define* S to be set of all $v \in B_1$ that send at least, say, $\frac{7}{16}n$ edges to B_2 (recall that $|B_2| \approx \frac{1}{2}n$). But even if we knew that $|S| \approx \delta n$ and were able to deal with $e(B_1, B_2)$, it would still be hard to control $e(B_1)$. The key to this problem is to prove that, as in the extremal example, there are (i) no edges at all from S to B_1 (see Fact 5.7.3 below) and (ii) no short odd cycles in B_1 (see Fact 5.7.1). The latter fact helps us in the light of Lemma 5.1 below.

Needless to say, many arguments occurring in this proof are inspired by [8]. On the other hand, even for $r = 2$ several new ideas are needed for going beyond (1.2).

§3. Coarse structure

Now we start to analyse the structure of K_{2r}-free graphs with huge minimum degree but without linear independent sets. The main result we shall obtain in this section reads as follows.

Proposition 3.1. *Given an integer $r \geqslant 2$ and a real $\eta > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for every K_{2r}-free graph G on $n \geqslant n_0$ vertices with $\alpha(G) < \delta n$ and $\delta(G) \geqslant \frac{3r-5}{3r-2}n$ there is a partition*

$$V(G) = A_1 \cupdot A_2 \cupdot \ldots \cupdot A_r$$

with the following properties:

- (i) *$e(A_i) \leqslant \eta n^2$ for all $i \in [r]$;*
- (ii) *if $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$, then $e(X_1, X_2) \leqslant \frac{1}{2}|X_1||X_2| + \eta n^2$.*

This will be shown by means of Szemerédi's famous regularity lemma [14] and we commence by introducing some terminology. Given a graph G and two nonempty disjoint sets $A, B \subseteq V(G)$ we say for two real numbers $\delta > 0$ and $d \in [0, 1]$ that the pair (A, B) is (δ, d)-*quasirandom* if for all $X \subseteq A$ and $Y \subseteq B$ the estimate $\big|e(X, Y) - d|X||Y|\big| \leqslant \delta|A||B|$ holds. If we just say that the pair (A, B) is δ-*quasirandom* we mean that it happens to be (δ, d)-quasirandom for $d = e(A, B)/|A||B|$.

Theorem 3.2 (Szemerédi's regularity lemma)**.** *Given $\xi > 0$ and $t_0 \in \mathbb{N}$ there exists an integer T_0 such that every graph G on $n \geqslant t_0$ vertices admits a partition*

$$V(G) = V_0 \cupdot V_1 \cupdot \ldots \cupdot V_t \tag{3.1}$$

of its vertex set such that

- *$t \in [t_0, T_0]$, $|V_0| \leqslant \xi|V(G)|$, and $|V_1| = \ldots = |V_t| > 0$,*
- *and for every $i \in [t]$ the set*

  $$\big\{j \in [t] \smallsetminus \{i\} \colon (V_i, V_j) \text{ is not } \xi\text{-quasirandom}\big\}$$

  has size at most ξt.

In the literature one often finds other versions of the regularity lemma, where instead of the second bullet above it is just demanded that at most ξt^2 pairs (V_i, V_j) with distinct $i, j \in [t]$ fail to be ξ-quasirandom. Applying such a regularity lemma to appropriate constants $\xi' \ll \xi$ and $t_0' \gg \max(t_0, \xi^{-1})$ and relocating partition classes with many irregular partners to V_0 one can obtain the version stated here; this argument has been used before by Łuczak [9], who explains it in more detail.

Next we deal with certain configurations in regular partitions of graphs with small independence number which allow us to build cliques. The lemma that follows is implicit in [3, Section 4] but for reasons of self-containment we shall supply its short proof. In its formulation we work with a one-sided version of quasirandomness that is enough for our purposes: If G is a graph, a pair (A, B) of disjoint subsets of $V(G)$ is said to be (δ, d)-*dense* for $\delta > 0$ and $d \in [0,1]$, if for all $X \subseteq A$ and $Y \subseteq B$ we have $e(X,Y) \geqslant d|X||Y| - \delta|A||B|$.

Lemma 3.3. *Suppose that integers $a \geqslant b \geqslant 1$ as well as a real number $\vartheta \in (0,1]$ are given and set $\xi = \left(\frac{\vartheta^2}{4}\right)^{a-1}$, $\delta = \left(\frac{\vartheta}{2}\right)^{a-1}$. Let H be a graph possessing a vertex partition*

$$H = V_1 \charfusion[\mathbin]{\cup}{\cdot} \ldots \charfusion[\mathbin]{\cup}{\cdot} V_a$$

into nonempty classes satisfying

- *(a) if $1 \leqslant i < j \leqslant a$, then (V_i, V_j) is (ξ, d_{ij})-dense for some $d_{ij} \in [\vartheta, 1]$;*
- *(b) if $1 \leqslant i < j \leqslant b$, then $d_{ij} \geqslant \frac{1}{2} + \vartheta$;*
- *(c) if $X \subseteq V_i$ and $|X| \geqslant \delta|V_i|$ for some $i \in [a]$, then X spans at least one edge in H.*

Then H contains a clique of order $a + b$.

Proof. We argue by induction on $a + b$. In the base case, $a = b = 1$, we have $\delta = 1$ and by condition (c) applied to $X = V_1$ there is indeed an edge in H.

In the induction step we certainly have $a \geqslant 2$ and we assume first that $a > b$. For every $i \in [a-1]$ the set

$$X(i) = \left\{v \in V_a \colon |N(v) \cap V_i| \leqslant \tfrac{\vartheta}{2}|V_i|\right\}$$

cannot be very large, as condition (a) yields

$$\tfrac{\vartheta}{2}|V_i||X(i)| \geqslant e(V_i, X(i)) \geqslant \vartheta|V_i||X(i)| - \xi|V_i||V_a|\,.$$

Together with $\xi \leqslant \frac{\vartheta}{2a}$ this leads to $|X(i)| \leqslant \frac{1}{a}|V_a|$. Now pick some $v_* \in V_a \smallsetminus \bigcup_{i \in [a-1]} X(i)$ and set $V_i' = N(v_*) \cap V_i$ for $i = 1, \ldots, a-1$. The definition of $X(i)$ gives $|V_i'| \geqslant \frac{\vartheta}{2}|V_i|$ for every $i \in [a-1]$ and, hence, the sets $V_1', \ldots, V_{a-1}'$ have the above properties (a), (b), and (c) for $a - 1$, $\frac{\xi}{\vartheta^2/4}$, and $\frac{\delta}{\vartheta/2}$ here in place of a, ξ, and δ there. So by the induction hypothesis the neighbourhood of v_* contains a K_{a+b-1}, wherefore indeed $K_{a+b} \subseteq H$.

The case $a = b$ is similar, but instead of the sets $X(i)$ introduced above we consider

$$Y(i) = \left\{v \in V_a \colon |N(v) \cap V_i| \leqslant \left(\tfrac{1}{2} + \tfrac{\vartheta}{2}\right)|V_i|\right\}$$

for $i \in [a-1]$. Invoking condition (*b*) one can show $|Y(i)| \leqslant \frac{1}{a}|V_a|$ in the same way as before and, hence, the set $L = V_a \smallsetminus \bigcup_{i\in[a-1]} Y(i)$ satisfies $|L| \geqslant \frac{1}{a}|V_a| \geqslant \delta|V_a|$. So by (*c*) there is an edge v_*w_* both of whose endvertices belong to L. Since $|N(v_*) \cap N(w_*) \cap V_i| \geqslant \vartheta|V_i|$ holds for each $i \in [a-1]$, the induction hypothesis allows us to find a K_{a+b-2} in the common neighbourhood of v_*w_* and again we obtain $K_{a+b} \subseteq H$. □

Suppose now that the regularity lemma has been applied, with a sufficiently small accuracy parameter ξ, to some graph G of small independence number, meaning that for some large integer t we have a partition of $V(G)$ such as (3.1). When one now attempts to find a K_{2r} in G by means of Lemma 3.3, it only matters which of the quasirandom pairs (V_i, V_j) have their densities, for an appropriate $\vartheta > 0$, in the interval $\left[\vartheta, \frac{1}{2} + \vartheta\right)$ or even in $\left[\frac{1}{2} + \vartheta, 1\right]$. We shall encode such information by the use of coloured edges in the reduced graph, with green edges corresponding to pairs that are either irregular or too sparse to be useful, and blue (or red) edges corresponding to quasirandom pairs of medium (or large) density.

Let us say that a *coloured graph* is a complete graph all of whose edges have been coloured red, blue, or green. Associated with any coloured graph G, say with vertex set V, we have its so-called *weight function* $w\colon V^2 \longrightarrow \{0, 1, 2\}$ defined by

$$w(x, y) = \begin{cases} 0 & \text{if } x = y \text{ or } xy \text{ is green,} \\ 1 & \text{if } xy \text{ is blue,} \\ 2 & \text{if } xy \text{ is red} \end{cases}$$

for all $x, y \in V$. We will often identify G with the pair (V, w). The *degree* of a vertex x of a coloured graph $G = (V, w)$ is defined to be the sum

$$d(x) = \sum_{y\in V} w(x, y)$$

and by $e(G)$ we mean half of the sum of the degrees $d(x)$ as x varies over V.

Two coloured graphs are said to be *isomorphic* if there is a colour-preserving bijection between their vertex sets. A coloured graph (V', w') is a *subgraph* of a coloured graph (V, w) if $V' \subseteq V$ and, additionally, $w'(x, y) \leqslant w(x, y)$ holds for all $x, y \in V'$.

Next, we come to the coloured graphs which are relevant in connection with Lemma 3.3. For integers $a \geqslant b \geqslant 1$ the coloured graph on a vertices without green edges whose red edges form a clique of order b will be denoted by $G_{a+b,b}$. For every integer $r \geqslant 2$ we set

$\mathscr{F}_{2r} = \{G_{2r,1}, \ldots, G_{2r,r}\}$. A coloured graph is said to be $\mathscr{F}_{2r}$-free if none of its subgraphs is isomorphic to a member of $\mathscr{F}_{2r}$.

In their proof of (1.1), Erdős, Hajnal, Szemerédi, and Sós use a lemma saying that every $\mathscr{F}_{2r}$-free coloured graph on n vertices satisfies $e(G) \leqslant \frac{3r-5}{3r-2}n^2$ (see [3, Lemma 3.3]). For the proof of Proposition 3.1 we will use a stability version of this lemma. There are various such statements, a rather strong one being the following.

Proposition 3.4. *Suppose that $r \geqslant 2$ and that G is a $\mathscr{F}_{2r}$-free coloured graph on n vertices with $\delta(G) > \frac{14r-24}{7r-5}n$. Then there is a partition $V(G) = W_1 \cup \ldots \cup W_r$ such that all edges within the partition classes are green and there are no red edges between W_1 and W_2.*

A somewhat lengthy proof of this result is given in [10]. For the purposes of the present work, however, it suffices to know only the weaker statement that follows. To keep this article as self-contained as possible, we will supply a quick sketch of its proof below.

Proposition 3.5. *Let $r \geqslant 2$ and let $\alpha > 0$ be sufficiently small. Then every $\mathscr{F}_{2r}$-free coloured graph G on n vertices with $\delta(G) \geqslant \frac{2(3r-5)-\alpha}{3r-2}n$ admits a partition*

$$V(G) = W_0 \cup W_1 \cup \ldots \cup W_r$$

of its vertex set such that $|W_0| \leqslant \alpha n$, all edges within the classes $W_1, \ldots, W_r$ are green, and no edge from W_1 to W_2 is red.

We prepare the proof of this proposition by the following variant of [3, Lemma 3.3], which can be proved in the same way. Let RK_{r-1} denote a red clique of order $r-1$ and set $\mathscr{F}_{2r}^{+} = \mathscr{F}_{2r} \cup \{RK_{r-1}\}$.

Lemma 3.6. *For $r \geqslant 2$ every $\mathscr{F}_{2r}^{+}$-free coloured graph G on n vertices satisfies*

$$e(G) \leqslant \tfrac{r-2}{r-1}n^2\,.$$

Proof. The case $r = 2$ is clear, for a RK_1 is nothing else than a vertex. So suppose $r \geqslant 3$ from now on. As in [3], two consecutive applications of Zykov's symmetrisation method [16] show that we may assume that there are is partition $V(G) = A_1 \cup \ldots \cup A_m$ and that for each $i \in [m]$ there is a partition $A_i = B_{i1} \cup \ldots \cup B_{ik_i}$ such that

- (i) for $i \in [m]$ and $j \in [k_i]$ all edges within B_{ij} are green;
- (ii) if $i \in [m]$ and $j, j' \in [k_i]$ are distinct, then all edges between B_{ij} and $B_{ij'}$ are blue;
- (iii) and for distinct $i, i' \in [m]$ all edges between A_i and $A_{i'}$ are red.

Since G contains neither RK_{r-1} nor $G_{2r,m}$, we have

$$1 \leqslant m \leqslant r-2 \qquad \text{and} \qquad k_1 + \ldots + k_m \leqslant 2r-1-m\,. \tag{3.2}$$

Set $\alpha_i = |A_i|/n$ for $i \in [m]$ and notice that $\sum_{i=1}^m \alpha_i = 1$. It is well known that (*i*) and (*ii*) imply $e(A_i) \leqslant \frac{k_i-1}{2k_i}|A_i|^2$ and thus it remains to prove

$$\sum_{1\leqslant i\leqslant m} \tfrac{k_i-1}{2k_i}\alpha_i^2 + 2\sum_{1\leqslant i<j\leqslant m} \alpha_i\alpha_j \leqslant \tfrac{r-2}{r-1}\,.$$

Subtracting this from $\big(\sum_{i=1}^m \alpha_i\big)^2 = 1$ we get

$$\sum_{i=1}^m \tfrac{k_i+1}{2k_i}\alpha_i^2 \geqslant \tfrac{1}{r-1}\,.$$

The Cauchy-Schwarz inequality yields

$$\sum_{i=1}^m \tfrac{k_i+1}{2k_i}\alpha_i^2 \cdot \sum_{i=1}^m \tfrac{2k_i}{k_i+1} \geqslant \Big(\sum_{i=1}^m \alpha_i\Big)^2 = 1$$

and thus it suffices to show that

$$\sum_{i=1}^m \tfrac{k_i}{k_i+1} \leqslant \tfrac{r-1}{2}\,. \tag{3.3}$$

Since the estimate $\frac{k}{k+1} \leqslant \frac{k+2}{6}$ holds for each positive integer k, it is enough to prove

$$\sum_{i=1}^m \tfrac{k_i+2}{6} \leqslant \tfrac{r-1}{2}$$

instead and in view of (3.2) this is clear. □

Proof of Proposition 3.5. Since $\frac{r-2}{r-1} < \frac{3r-5}{3r-2}$ and $\alpha \ll 1$, we may suppose that $e(G) > \frac{r-2}{r-1}$. By Lemma 3.6 and the assumption that G be $\mathscr{F}_{2r}$-free it follows that G contains a RK_{r-1}, say with vertex set $K = \{v_1, v_3, \dots v_r\}$. The minimum degree condition and $\alpha \ll 1$ yield

$$\sum_{x\in V(G)} \big(2r-2-d_K(x)\big) = \sum_{v\in K}\big(2n-d(v)\big) \leqslant \tfrac{(6+\alpha)(r-1)}{3r-2}n < 2n$$

and, hence, there is a vertex $v_2 \in V(G)$ with $2r-2-d_K(v_2) \leqslant 1$. As G contains no $G_{2r,r} = RK_r$, it follows that v_2 has exactly one blue neighbour in K and sends red edges to all other members of K. By symmetry we may suppose that v_1v_2 is blue. Set

- $L = \{v_1, \dots, v_r\} = K \cup \{v_2\}$,
- $W_i = \big\{x \in V(G)\colon$ if $j \in [r]$, then $w(x, v_j) = w(v_i, v_j)\big\}$ for $i = 1, \dots, r$,
- $W_0 = V(G) \smallsetminus (W_1 \cup \dots \cup W_r)$,
- and $q(x) = 2(3r-2) - 2\big(w(v_1,x) + w(v_2,x)\big) - 3\big(w(v_3,x) + \dots + w(v_r,x)\big)$ for every $x \in V(G)$.

Notice that the sets $W_1, \dots, W_r$ are mutually disjoint. Exploiting that G contains neither $G_{2r,r}$ nor $G_{2r,r-1}$ one checks easily that

- all edges within one of the partition classes $W_1, \dots, W_r$ are green
- no edge from W_1 to W_2 is red,

- $q(x) \geqslant 6$ for all $x \in V(G)$,
- and that equality holds in the previous bullet if and only if $x \in W_1 \cup \ldots \cup W_r$.

It remains to show that $|W_0| \leqslant \alpha n$. To this end we write

$$|W_0| \leqslant \sum_{x \in V(G)} (q(x) - 6) = 2(3r-5)n - 2\big(d(v_1) + d(v_2)\big) + 3\big(d(v_3) + \ldots + d(v_r)\big)$$

and apply the minimum degree condition again. □

Finally, we show the main result of this section.

Proof of Proposition 3.1. Take appropriate constants

$$\delta \ll T_0^{-1} \ll t_0^{-1}, \xi \ll \vartheta \ll \min(\eta, r^{-1}),$$

where T_0 is obtained by applying the regularity lemma to t_0 and ξ, and set $n_0 = t_0$. Consider a K_{2r}-free graph G on $n \geqslant n_0$ vertices with $\alpha(G) < \delta n$ and $\delta(G) \geqslant \frac{3r-5}{3r-2}n$. The regularity lemma yields for some integers $t \in [t_0, T_0]$ and $m \geqslant 1$ a partition

$$V(G) = V_0 \cup V_1 \cup \ldots \cup V_t$$

such that $|V_0| \leqslant \xi n$, $|V_1| = \ldots = |V_t| = m$, and for every $i \in [t]$ all but at most ξt indices $j \in [t] \smallsetminus \{i\}$ have the property that (V_i, V_j) is ξ-quasirandom.

Define a coloured graph H with vertex set $[t]$ by declaring a pair ij to be *green* if (V_i, V_j) either fails to be ξ-quasirandom or has a density smaller than ϑ, *blue* if (V_i, V_j) is ξ-quasirandom and has a density in $\big[\xi, \frac{1}{2} + \xi\big)$, and *red* otherwise.

As a consequence of Lemma 3.3, H is $\mathscr{F}_{2r}$-free. Next, we will show that

$$\delta(H) \geqslant 2\big(\tfrac{3r-5}{3r-2} - 3\vartheta\big)t\,. \tag{3.4}$$

To verify this, we consider an arbitrary vertex i of H and denote the numbers of its blue and red neighbours by a and b, respectively. The minimum degree condition on G yields

$$\tfrac{3r-5}{3r-2}mn \leqslant \sum_{j=0}^{t} e(V_i, V_j)\,.$$

On the right side of this estimate, the term corresponding to $j = 0$ contributes at most ξmn, $j = i$ contributes at most m^2, and the irregular pairs contribute at most $\xi t m^2$. Consequently we have

$$\big(\tfrac{3r-5}{3r-2} - \xi\big)mn \leqslant m^2 + \xi t m^2 + t\vartheta m^2 + a\big(\tfrac{1}{2} + \vartheta\big)m^2 + bm^2.$$

Using $n \geqslant mt$ and canceling m^2 we infer

$$\big(\tfrac{3r-5}{3r-2} - \xi\big)t \leqslant (2\vartheta + \xi)t + 1 + \tfrac{1}{2}d_H(i)\,.$$

So in view of $t \geqslant t_0 \gg \vartheta^{-1}$ and $\xi \ll \vartheta$ we obtain $d_H(i) \geqslant 2\big(\frac{3r-5}{3r-2} - 3\vartheta\big)t$, which proves (3.4).

By Proposition 3.5 and $\vartheta \ll r^{-1}$ there exists a partition

$$[t] = W_0 \charfusion[\mathbin]{\cup}{\cdot} W_1 \charfusion[\mathbin]{\cup}{\cdot} \dots \charfusion[\mathbin]{\cup}{\cdot} W_r$$

such that $|W_0| \leqslant 18\vartheta rt$, all edges within $W_1, \dots, W_t$ are green, and no edge between W_1 and W_2 is red. For $s \in [0, r]$ we define

$$A_s^* = \bigcup_{i \in W_s} V_i \,.$$

Then $V(G) = V_0 \cup A_0^* \cup A_1^* \cup \dots \cup A_r^*$ is a partition of $V(G)$ and

$$|V_0| + |A_0^*| \leqslant \xi n + |W_0|m \leqslant (\xi + 18r\vartheta)n \leqslant \tfrac{1}{2}\eta n \,.$$

This means that if we manage to show

(a) $e(A_s^*) \leqslant \frac{1}{2}\eta n^2$ for all $s \in [r]$,

(b) and $e(X_1, X_2) \leqslant \frac{1}{2}|X_1||X_2| + \frac{1}{2}\eta n^2$ for all $X_1 \subseteq A_1^*$ and $X_2 \subseteq A_2^*$,

then the partition $V(G) = A_1 \cup \dots \cup A_r$ defined by $A_1 = V_0 \cup A_0^* \cup A_1^*$ and $A_s = A_s^*$ for $s \in [2, r]$ has both desired properties.

To prove (a) we start for a given $s \in [r]$ from the decomposition

$$e(A_s^*) = \sum_{i \in W_s} e(V_i) + \sum_{ij \in W_s^{(2)}} e(V_i, V_j) \,.$$

Here, each of the at most t terms in the first sum is at most $m^2/2$. Besides, there are at most $\xi t^2/2$ terms corresponding to irregular pairs in the second sum, and each of them amounts to no more than m^2. Finally, the remaining at most $t^2/2$ terms in the second sum correspond to pairs whose density is at most ϑ. Thus we obtain

$$e(A_s^*) \leqslant \big(\tfrac{1}{2t} + \tfrac{\xi}{2} + \tfrac{\vartheta}{2}\big)m^2 t^2$$

and due to $t \geqslant t_0$ and $mt \leqslant n$ an appropriate choice of our constants does indeed guarantee that $e(A_s^*) \leqslant \frac{1}{2}\eta n^2$.

Similarly, the proof of (b) employs

$$e(X_1, X_2) = \sum_{i \in W_1} \sum_{j \in W_2} e(V_i \cap X_1, V_j \cap X_2) \,.$$

Again the contribution caused by irregular pairs is at most $\xi n^2/2$. The remaining terms correspond to ξ-quasirandom pairs, which owing to the absence of red edges from W_1 to W_2 have density at most $\frac{1}{2} + \vartheta$. Consequently,

$$\begin{aligned}
e(X_1, X_2) &\leqslant \sum_{i \in W_1} \sum_{j \in W_2} \Big[\big(\tfrac{1}{2} + \vartheta\big)|V_i \cap X_1||V_j \cap X_2| + \xi|V_i||V_j|\Big] + \tfrac{1}{2}\xi n^2 \\
&\leqslant \big(\tfrac{1}{2} + \vartheta\big)|X_1||X_2| + \xi t^2 m^2 + \tfrac{1}{2}\xi n^2 \\
&\leqslant \tfrac{1}{2}|X_1||X_2| + \big(\vartheta + \tfrac{3}{2}\xi\big)n^2 \leqslant \tfrac{1}{2}|X_1||X_2| + \tfrac{1}{2}\eta n^2
\end{aligned}$$

and the proof of Proposition 3.1 is complete. $\square$

§4. EXACT PARTITIONS

4.1. **More information.** It turns out that the lower bound $e(G) \geqslant \frac{3r-5}{3r-2} \cdot \frac{n^2}{2}$, which follows from the minimum degree condition in Proposition 3.1, gives us further information on the sizes of the vertex classes of the partition obtained there and on the edge densities between these classes. This happens due to the following elementary inequality.

Lemma 4.1. *If for $r \geqslant 2$ the real numbers $a_1, \ldots, a_r$ sum up to 1, then*

$$\sum_{1 \leqslant i < j \leqslant r} a_i a_j - \tfrac{1}{2} a_1 a_2 \leqslant \tfrac{3r-5}{2(3r-2)} \,.$$

Moreover, if for some real $\varrho \geqslant 0$ we have

$$\sum_{1 \leqslant i < j \leqslant r} a_i a_j - \tfrac{1}{2} a_1 a_2 \geqslant \tfrac{3r-5}{2(3r-2)} - \varrho \,, \tag{4.1}$$

then $\left|a_i - \frac{2}{3r-2}\right| \leqslant 2\sqrt{\varrho}$ for $i = 1, 2$ and $\left|a_i - \frac{3}{3r-2}\right| \leqslant 2\sqrt{\varrho}$ for $i = 3, \ldots, r$.

Proof. Define

$$\alpha_i = \begin{cases} a_i - \frac{2}{3r-2} & \text{if } i = 1, 2 \\ a_i - \frac{3}{3r-2} & \text{if } i = 3, \ldots, r \end{cases}$$

and observe that

$$\sum_{i=1}^{r} \alpha_i^2 + \alpha_1 \alpha_2 = \sum_{i=1}^{r} a_i^2 + a_1 a_2 - \sum_{i=1}^{r} \frac{6a_i}{3r-2} + \frac{4 \cdot 3 + 9(r-2)}{(3r-2)^2} = \sum_{i=1}^{r} a_i^2 + a_1 a_2 - \frac{3}{3r-2} \,.$$

Due to $\left(\sum_{i=1}^{r} a_i\right)^2 = 1$ this rewrites as

$$\tfrac{1}{2}\alpha_1^2 + \tfrac{1}{2}\alpha_2^2 + \tfrac{1}{2}(\alpha_1 + \alpha_2)^2 + \sum_{i=3}^{r} \alpha_i^2 \leqslant \tfrac{3r-5}{3r-2} - \Big(2\sum_{i<j} a_i a_j - a_1 a_2\Big) \,,$$

which establishes the first part of our claim. Moreover, if (4.1) holds for some $\varrho \geqslant 0$ we obtain

$$\tfrac{1}{2}\alpha_1^2 + \tfrac{1}{2}\alpha_2^2 + \tfrac{1}{2}(\alpha_1 + \alpha_2)^2 + \sum_{i=3}^{r} \alpha_i^2 \leqslant 2\varrho \,,$$

whence $|\alpha_i| \leqslant 2\sqrt{\varrho}$ holds for all $i \in [r]$. $\square$

With this lemma at hand we may prove the following estimates.

Fact 4.2. *Suppose that a graph G and the partition*

$$V(G) = A_1 \cupdot \ldots \cupdot A_r$$

are as described and obtained in Proposition 3.1. Then

- $\left||A_i| - \frac{2n}{3r-2}\right| \leqslant 2\sqrt{(r+1)\eta} \cdot n$ *for $i = 1, 2$,*

- $\big||A_i| - \frac{3n}{3r-2}\big| \leqslant 2\sqrt{(r+1)\eta}\cdot n$ *for* $i = 3, \dots, r$,
- $e(A_1, A_2) \geqslant \frac{1}{2}|A_i||A_j| - r\eta n^2$,
- *and* $e(A_i, A_j) \geqslant |A_i||A_j| - (r+1)\eta n^2$ *whenever* $1 \leqslant i < j \leqslant n$ *and* $(i, j) \neq (1, 2)$.

Proof. The minimum degree condition $\delta(G) \geqslant \frac{3r-5}{3r-2}n$ yields $e(G) \geqslant \frac{3r-5}{3r-2}\cdot\frac{n^2}{2}$ and due to Proposition 3.1(*i*) it follows that

$$\begin{aligned}\Big(\tfrac{3r-5}{2(3r-2)} - (r+1)\eta\Big)n^2 + \sum_{\substack{1\leqslant i<j\leqslant r\\(i,j)\neq(1,2)}} \big[|A_i||A_j| - e(A_i, A_j)\big] + \big[\tfrac{1}{2}|A_1||A_2| + \eta n^2 - e(A_1, A_2)\big]&\\ \leqslant \sum_{1\leqslant i<j\leqslant r} |A_i||A_j| - \tfrac{1}{2}|A_1||A_2|\,.&\qquad(4.2)\end{aligned}$$

The square brackets on the left side being positive we deduce

$$\Big(\tfrac{3r-5}{2(3r-2)} - (r+1)\eta\Big)n^2 \leqslant \sum_{1\leqslant i<j\leqslant r} |A_i||A_j| - \tfrac{1}{2}|A_1||A_2|$$

and the case $\varrho = (r+1)\eta$ of Lemma 4.1 leads to the first two bullets.

Furthermore, Lemma 4.1 provides an upper bound of $\frac{3r-5}{3r-2}\cdot\frac{n^2}{2}$ on the right side of (4.2). Therefore we have

$$\sum_{\substack{1\leqslant i<j\leqslant r\\(i,j)\neq(1,2)}} \big[|A_i||A_j| - e(A_i, A_j)\big] + \big[\tfrac{1}{2}|A_1||A_2| + \eta n^2 - e(A_1, A_2)\big] \leqslant (r+1)\eta n^2\,.$$

and the last two bullets follow as well. □

4.2. **Local minimum degree.** Along the way leading from the partition provided by Proposition 3.1 to our main theorem we will need to make further efficient uses of the assumption $K_{2r} \nsubseteq G$. It should be clear that *building* a K_{2r} in G would be easier if we knew that certain minimum degree conditions hold between the partition classes and the goal of this section is to enforce several such conditions by moving a few vertices violating them to other classes into which they fit better. For later reference we include the somewhat lengthy list of properties that we shall obtain into a definition.

Definition 4.3. Let an integer $r \geqslant 2$, a real $\varepsilon > 0$, an n-vertex graph G, and a partition

$$V(G) = B_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} B_r$$

be given. Set $d_i(v) = d_{B_i}(v)$ for all $v \in V(G)$ and $i \in [r]$. We say that the above partition is (r, ε)-*exact* if the following conditions hold.

(α) For $i = 1, 2$ one has $\big||B_i| - \frac{2n}{3r-2}\big| \leqslant \varepsilon n$.
(β) For $i = 3, \dots, r$ one has $\big||B_i| - \frac{3n}{3r-2}\big| \leqslant \varepsilon n$.
(γ) If $i \in [r]$, then $e(B_i) \leqslant \varepsilon n^2$.
(δ) If $X_1 \subseteq B_1$ and $X_2 \subseteq B_2$, then $\big|e(X_1, X_2) - \frac{1}{2}|X_1||X_2|\big| \leqslant \varepsilon n^2$.
(ε) If $1 \leqslant i < j \leqslant r$ and $(i, j) \neq (1, 2)$, then $e(B_i, B_j) \geqslant |B_i||B_j| - \varepsilon n^2$.

- (ζ) If $\{i, j\} = \{1, 2\}$ and $v \in B_i$, then $d_j(v) \geqslant \frac{1/3}{3r-2}n$.
- (η) If $i \in \{1, 2\}$, $j \in [3, r]$, and $v \in B_i$, then $d_j(v) \geqslant \frac{5/3}{3r-2}n$.
- (ϑ) If $i \in [3, r]$, $j \in \{1, 2\}$, and $v \in B_i$, then $d_j(v) \geqslant \frac{1/5}{3r-2}n$.
- (ι) If $i, j \in [3, r]$ are distinct and $v \in B_i$, then $d_j(v) \geqslant \frac{1}{3r-2}n$.

The main result of this subsection is the following.

Proposition 4.4. *For every $r \geqslant 2$ and $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that every K_{2r}-free graph G on $n \geqslant n_0$ vertices, with $\delta(G) \geqslant \frac{3r-5}{3r-2}n$ and $\alpha(G) < \delta n$ has an (r, ε)-exact partition.*

Proof. By monotonicity we may assume that ε is sufficiently small so that all estimates to be performed below will hold. We commence be choosing a sufficiently small $\eta \ll \varepsilon$. With this number η we appeal to Proposition 3.1 and it answers with an integer $n_0 \in \mathbb{N}$ and with some $\delta > 0$. We claim that these two constants have the desired properties.

Let any K_{2r}-free graph G on $n \geqslant n_0$ vertices with $\alpha(G) < \delta n$ and $\delta(G) \geqslant \frac{3r-5}{3r-2}n$ be given and take a partition

$$V(G) = A_1^0 \cup A_2^0 \cup \ldots \cup A_r^0 \tag{4.3}$$

such that

- (i) $e(A_i^0) \leqslant \eta n^2$ for all $i \in [r]$;
- (ii) if $X_1 \subseteq A_1^0$ and $X_2 \subseteq A_2^0$, then $e(X_1, X_2) \leqslant \frac{1}{2}|X_1||X_2| + \eta n^2$.

Due to Fact 4.2 and $\eta \ll \varepsilon$ we may suppose moreover that

- (iii) for $i = 1, 2$ we have $\big||A_i^0| - \frac{2n}{3r-2}\big| \leqslant \frac{1}{2}\varepsilon n$;
- (iv) for $i = 3, \ldots, r$ we have $\big||A_i^0| - \frac{3n}{3r-2}\big| \leqslant \frac{1}{2}\varepsilon n$;
- (v) $e(A_1^0, A_2^0) \geqslant \frac{1}{2}|A_1^0||A_2^0| - \frac{1}{4}\varepsilon n^2$;
- (vi) and that $e(A_i^0, A_i^0) \geqslant |A_i^0||A_j^0| - \frac{1}{2}\varepsilon n^2$ whenever $1 \leqslant i < j \leqslant r$ and $(i, j) \neq (1, 2)$.

We need to define an (r, ε)-exact partition of G. To this end we perform a recursive procedure, in the course of which a sequence of partitions of $V(G)$ into r parts is constructed. The starting point is (4.3). In each step only one vertex is moved from one vertex class to another one, while all other vertices stay in the partition class they have belonged to before. Let

$$V(G) = A_1^s \cup A_2^s \cup \ldots \cup A_r^s$$

be the partition that we have after s steps and put

$$\Omega_s = 6e(A_1^s) + 6e(A_2^s) + \sum_{i=3}^{r} e(A_i^s)\,.$$

When the s^{th} step is to carried out, we ensure that

$$\Omega_s \leqslant \Omega_{s-1} - \frac{1/4}{3r-2}n \tag{4.4}$$

holds. This condition guarantees inductively that $\Omega_s \leqslant \Omega_0 - \frac{s/4}{3r-2}n$ and because of $\Omega_s \geqslant 0$ this means that at some moment we will run out of permissible steps. When this happens we stop the procedure and we let

$$V(G) = B_1 \cupdot B_2 \cupdot \ldots \cupdot B_r \tag{4.5}$$

be the terminal partition. The remainder of this proof is dedicated to proving that this partition is (r, ε)-exact. If the above procedure lasted for t steps, then

$$\tfrac{t/4}{3r-2}n \leqslant \Omega_0 \overset{(i)}{\leqslant} (r+10)\eta n^2$$

informs us that

$$t \leqslant 4(3r-2)(r+10)\eta n \leqslant 48r^2\eta n\,. \tag{4.6}$$

In particular, $\eta \ll \varepsilon \ll 1$ allows us to conclude that $t \leqslant \frac{1}{2}\varepsilon n$. Since only t vertices were moved during the process, it follows from this bound and from *(iii)* as well as *(iv)* that the clauses (α) and (β) of Definition 4.3 are satisfied.

For fixed $i \in [r]$ the current value of $e(A_i)$ can change by at most n in every step and thus we have

$$e(B_i) \leqslant e(A_i^0) + tn \leqslant 49r^2\eta n^2 \leqslant \varepsilon n^2$$

by *(i)* and (4.6), which shows the validity of (γ). The proof of (ε) is very similar but uses *(vi)* instead of *(i)*. We leave the details to the reader. Proceeding similarly with *(v)* one can obtain

$$e(B_1, B_2) \geqslant \tfrac{1}{2}|B_1||B_2| - \tfrac{1}{2}\varepsilon n^2\,. \tag{4.7}$$

Let us continue with (δ). For any two sets $X_1 \subseteq B_1$ and $X_2 \subseteq B_2$ we have

$$\begin{aligned} e(X_1, X_2) &\leqslant e(X_1 \cap A_1^0, X_2, \cap A_2^0) + (|B_1 \smallsetminus A_1^0| + |B_2 \smallsetminus A_2^0|)n \\ &\overset{(ii)}{\leqslant} \tfrac{1}{2}|X_1 \cap A_1^0||X_2, \cap A_2^0| + \eta n^2 + tn \end{aligned}$$

and in view of (4.6) it follows that

$$e(X_1, X_2) \leqslant \tfrac{1}{2}|X_1||X_2| + \tfrac{1}{4}\varepsilon n^2\,. \tag{4.8}$$

We still need an estimate in the other direction and for this purpose we invoke (4.7) and make two applications of (4.8), thus getting

$$\begin{aligned} e(X_1, X_2) &= e(B_1, B_2) - e(B_1, B_2 \smallsetminus X_2) - e(B_1 \smallsetminus X_1, X_2) \\ &\geqslant \big(\tfrac{1}{2}|B_1||B_2| - \tfrac{1}{2}\varepsilon n^2\big) - \big(\tfrac{1}{2}|B_1||B_2 \smallsetminus X_2| + \tfrac{1}{4}\varepsilon n^2\big) - \big(\tfrac{1}{2}|B_1 \smallsetminus X_1||X_2| + \tfrac{1}{4}\varepsilon n^2\big) \\ &= \tfrac{1}{2}|X_1||X_2| - \varepsilon n^2\,. \end{aligned}$$

Altogether the pair (B_1, B_2) behaves indeed as demanded by (δ).

It remains to deal with the local minimum conditions (ζ), (η), (ϑ), and (ι). The proofs of all four of them are very similar and rely on the property (4.4) of the procedure that was used to generate the partition (4.5). We will only display the proof (η) here and leave the three other clauses to the reader.

Assume, for instance, that there is a vertex $v \in B_1$ with $d_3(v) < \frac{5/3}{3r-2}n$. Due to the minimum degree condition imposed on G we must have

$$d_1(v) \geqslant \tfrac{3r-5}{3r-2}n - |B_2| - \tfrac{5/3}{3r-2}n - \sum_{i=4}^{r} |B_i|\,.$$

Because of (α) and (β) this implies

$$d_1(v) \geqslant \tfrac{1/3}{3r-2}n - (r-2)\varepsilon n\,,$$

wherefore

$$6d_1(v) - d_3(v) > \tfrac{1/4}{3r-2}n\,.$$

Consequently we can perform a $(t+1)^{\text{st}}$ step of our procedure and move v from B_1 to B_3. This contradicts the supposed maximality of t, and thereby (η) is proved. □

§5. Refined edge counting

Let us start this section with an elementary lemma, the following.

Lemma 5.1. *Every graph G not containing a cycle of length 3, 5, or 7 satisfies*

$$e(G) \leqslant \alpha(G)^2\,.$$

Proof. We construct recursively a sequence $z_1, \dots, z_k$ of distinct vertices of G according to the following rules.

- Let z_1 be any vertex of G whose degree is maximal.
- If at some moment the vertices $z_1, \dots, z_i$ have already been selected, we ask ourselves whether the set Q_i of all vertices having a distance of at least four from all of them is empty or not.
- If $Q_i = \varnothing$, we set $k = i$ and terminate the procedure.
- Otherwise we take a vertex $z_{i+1} \in Q_i$ whose degree is as large as possible.

Set $Q_0 = V(G)$ and $W_i = Q_{i-1} \smallsetminus Q_i$ for $i = 1, \dots, k$. Notice that

$$V(G) = W_1 \cupdot \dots \cupdot W_k$$

is indeed a partition, because $Q_0 \supseteq Q_1 \supseteq \dots \supseteq Q_k = \varnothing$. Owing to the maximum degree conditions imposed on the vertices z_i we have

$$2e(G) = \sum_{x \in V(G)} d(x) \leqslant \sum_{i=1}^{k} |W_i| \cdot d(z_i)\,. \tag{5.1}$$

We contend that for $i \in [k]$ every vertex $x \in W_i$ has at most distance three from z_i. To see this we remark that due to $x \notin Q_i$ there has to be an index $j \in [i]$ such that x has distance at most three from z_j. Moreover, $j < i$ would yield $x \notin Q_{i-1}$, contrary to $x \in W_i$. Thus we must have $j = i$, as desired.

It follows that we can partition W_i into a set of vertices having distance 0 or 2 from z_i and a set of vertices having distance 1 or 3 from z_i. Both partition classes are independent sets, for otherwise G would contain an odd cycle of length 3, 5, or 7.

In particular, we have $|W_i| \leqslant 2\alpha(G)$ for each $i \in [k]$ and in view of (5.1) we obtain

$$e(G) \leqslant \alpha(G) \sum_{i=1}^{k} d(z_i) \,.$$

Due to their construction any two of the vertices $z_1, \dots, z_k$ have a distance of at least four. Therefore, their neighbourhoods are mutually disjoint and taken together they form an independent set. Thus we have indeed $e(G) \leqslant \alpha(G)^2$. □

After this little distraction we resume our task of proving Theorem 1.2. In the light of the work in the two previous sections, it seems desirable to deal with the case that G admits an exact partition, which will occupy the remainder of the present section.

Proposition 5.2. *Given an integer $r \geqslant 2$, there exists a real $\varepsilon > 0$ such that for every $\delta \leqslant \varepsilon$ every n-vertex graph G with $K_{2r} \nsubseteq G$ and $\alpha(G) \leqslant \delta n$ admitting an (r, ε)-exact partition of its vertex set has at most $\left(\frac{3r-5}{3r-2} + \delta - \delta^2\right)\frac{n^2}{2}$ edges.*

Proof. Throughout the arguments that follow we will assume that ε has been chosen so small that all estimates encountered below hold. Now let $\delta \leqslant \varepsilon$, let $G = (V, E)$ be a K_{2r}-free graph on n vertices with $\alpha(G) < \delta n$ and let

$$V = B_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} B_r$$

be an (r, ε)-exact partition of G. By lowercase greek letters enclosed in parentheses such as (α), …, (ι) we shall always mean the corresponding clauses of Definition 4.3.

The statement that follows will often be useful in conjunction with the hypothesis that G be K_{2r}-free.

Claim 5.3. *Suppose that $I \subseteq [r]$ and that for every $i \in I$ we have a set $X_i \subseteq B_i$ with $|X_i| \geqslant \frac{1/15}{3r-2} n$. Then the set $X = \bigcup_{i \in I} X_i$ contains a clique of order $2|I| - 1$.*

Moreover, if I does not contain both of 1 and 2, than X does even contain a clique of order $2|I|$.

Proof. Let us begin with the "moreover"-part. Intending to apply Lemma 3.3 with $\vartheta = \frac{1}{2}$ and $a = b = |I|$ we need to check that for distinct $i, j \in I$ the pair (X_i, X_j) is $(16^{-r}, 1)$-dense and that $\alpha(G) < |X_i|/4^r$. The latter is an immediate consequence of $\delta \leqslant \varepsilon \ll 1$. Moreover, if $Y_i \subseteq X_i$ and $Y_j \subseteq X_j$, then

$$e(Y_i, Y_j) \overset{(\varepsilon)}{\geqslant} |Y_i||Y_j| - \varepsilon n^2 \geqslant |Y_i||Y_j| - 16^{-r}|X_i||X_j|\,,$$

as desired. If $1, 2 \in I$ we can still apply Lemma 3.3 with $\vartheta = \frac{1}{2}$, but this time with $a = |I|$ and $b = |I| - 1$. This is because (δ) allows us to show, in the same way as above, that the pair (X_1, X_2) is $(1/16^r, 1/2)$-dense. □

Next we explain how condition (γ) is utilised.

Claim 5.4. *If $i \in [r]$ and $X \subseteq B_i$, then $e(X) \leqslant \frac{n/60}{3r-2}|X|$.*

Proof. If $|X| \leqslant \frac{n/60}{3r-2}$ this follows from the trivial bound $e(X) \leqslant |X|^2$. On the other hand, if $|X| \geqslant \frac{n/60}{3r-2}$, then we have

$$e(X) \overset{(\gamma)}{\leqslant} \varepsilon n^2 \leqslant \left(\tfrac{n/60}{3r-2}\right)^2 \leqslant \tfrac{n/60}{3r-2}|X|$$

due to $\varepsilon \ll 1$. □

Claim 5.5. *For each $i \in [3, r]$ we have*

$$e(B_i, V) \leqslant (n - |B_i|)|B_i| + \delta n|B_i|\,.$$

Proof. Look at the partition $B_i = B_i^+ \charfusion[\mathbin]{\cup}{\cdot} B_i^-$ defined by

$$B_i^+ = \left\{x \in B_i \colon |N(x) \smallsetminus B_i| \geqslant n - |B_i| - \tfrac{n/15}{3r-2}\right\}$$

and $B_i^- = B_i \smallsetminus B_i^+$. Clearly, we have

$$e(B_i, V \smallsetminus B_i) \leqslant (n - |B_i|)|B_i| - \tfrac{n/15}{3r-2}|B_i^-| \tag{5.2}$$

and Claim 5.4 yields

$$e(B_i^-) \leqslant \tfrac{n/60}{3r-2}|B_i^-|\,. \tag{5.3}$$

Now assume for the sake of contradiction that B_i contains a triangle uvw two of whose vertices, say v and w, belong to B_i^+. Let X denote the common neighbourhood of u, v, and w. The definition of B_i^+ leads to

$$|X \cap B_j| \geqslant |N(u) \cap B_j| - \tfrac{2/15}{3r-2}n \overset{(\iota)}{\geqslant} \tfrac{13/15}{3r-2}n$$

for $j \in [3, r] \smallsetminus \{i\}$ and, similarly, we have $|X \cap B_j| \geqslant \frac{1/15}{3r-2}n$ for $j = 1, 2$ due to (ϑ). Thus the assumptions of Claim 5.3 are satisfied by $I = [r] \smallsetminus \{i\}$ and X, meaning that X contains a K_{2r-3}. But together with the triangle uvw this clique gives us a K_{2r} in G, which is absurd.

This proves that there are no such triangles in B_i and due to $\alpha(G) < \delta n$ it follows that no vertex in B_i can have more than δn neighbours in B_i^+. Therefore we have

$e(B_i^+, B_i^-) \leqslant \delta n|B_i^-|$ and $2e(B_i^+) \leqslant \delta n|B_i^+|$. Taking (5.2) and (5.3) into account we can now deduce

$$\begin{aligned} e(B_i, V) &= e(B_i, V \smallsetminus B_i) + 2e(B_i^+) + 2e(B_i^+, B_i^-) + 2e(B_i^-) \\ &\leqslant (n - |B_i|)|B_i| + \delta n|B_i| + \left(\tfrac{n/30}{3r-2} + \delta n - \tfrac{n/15}{3r-2}\right)|B_i^-|\,, \end{aligned}$$

and in view of $\delta \ll 1$ the desired estimate follows. □

Before we proceed deriving similar upper bounds for $e(B_1, V)$ and $e(B_2, V)$, we record some useful properties of the common neighbourhoods of edges in B_1.

Claim 5.6. *Any two vertices $u, v \in B_1$ forming an edge have at least $\frac{4/15}{3r-2}n$ common neighbours in each of $B_3, \dots, B_r$, but less than $\frac{1/15}{3r-2}n$ common neighbours in B_2.*

Proof. For each $i \in [3, r]$ we have

$$|N(u) \cap N(v) \cap B_i| \geqslant |N(u) \cap B_i| + |N(v) \cap B_i| - |B_i|\,,$$

which due to (β) and (η) yields

$$|N(u) \cap N(v) \cap B_i| \geqslant \tfrac{10/3}{3r-2}n - \left(\tfrac{3}{3r-2} + \varepsilon\right)n \geqslant \tfrac{4/15}{3r-2}n\,,$$

as desired. If u and v had at least $\frac{1/15}{3r-2}n$ common neighbours in B_2, we could use Claim 5.3 with $I = [r] \smallsetminus \{1\}$ to find a K_{2r-2} among the common neighbours of those two vertices, contrary to $K_{2r} \nsubseteq G$. □

Claim 5.7. *For $i \in \{1, 2\}$ we have*

$$e(B_i, V) \leqslant |B_i|(n - |B_1| - |B_2|) + \tfrac{1}{2}|B_1||B_2| + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2\,.$$

Proof. Due to symmetry it suffices to prove this for $i = 1$ only. The vertices in

$$P = \left\{x \in B_1 \colon |N(x) \smallsetminus B_1| \leqslant n - |B_1| - \tfrac{1}{2}|B_2| - \tfrac{1/15}{3r-2}n\right\} \tag{5.4}$$

receive special treatment.

Fact 5.7.1. *There is no triangle in B_1 two of whose vertices are outside P.*

Proof. Arguing indirectly we assume that uvw is such a triangle. By Claim 5.6 no two of the three vertices u, v, and w can have $\frac{1/15}{3r-2}n$ common neighbours in B_2, whence

$$d_2(u) + d_2(v) + d_2(w) < |B_2| + \tfrac{1/5}{3r-2}n\,.$$

On the other hand, by the definition of P we have $d_2(x) > \frac{1}{2}|B_2| - \frac{1/15}{3r-2}n$ for every $x \in B_1 \smallsetminus P$ and together with (ζ) this yields

$$d_2(u) + d_2(v) + d_2(w) > 2\left(\tfrac{1}{2}|B_2| - \tfrac{1/15}{3r-2}n\right) + \tfrac{1/3}{3r-2}n = |B_2| + \tfrac{1/5}{3r-2}n\,.$$

This contradiction proves Fact 5.7.1. □

Since $\alpha(G) < \delta n$, it follows that no vertex in P can have δn neighbours in $B_1 \setminus P$, which in turn reveals $e(P, B_1 \setminus P) \leqslant \delta n|P|$. Together with the estimate $e(P) \leqslant \frac{n/60}{3r-2}|P|$, which follows from Claim 5.4, this gives

$$2e(P, B_1 \setminus P) + 2e(P) \leqslant \big(2\delta + \tfrac{1/30}{3r-2}\big)n|P| \leqslant \tfrac{1/15}{3r-2}n|P|$$

and by adding the upper bound on $e(P, V \setminus B_1)$ that trivially follows from (5.4) we arrive at

$$e(P, V) + e(P, B_1 \setminus P) \leqslant |P|(n - |B_1| - \tfrac{1}{2}|B_2|)\,. \tag{5.5}$$

Fact 5.7.2. *There is no C_3, C_5, or C_7 in G all of whose vertices are in $B_1 \setminus P$.*

Proof. Assume contrariwise that for some $\ell \in \{3, 5, 7\}$ the vertices in $C = \{v_1, \dots, v_\ell\}$ form such a cycle. If a vertex $x \in B_2$ is adjacent to q vertices in C, then the neighbourhood of x contains at least $q - \frac{1}{2}(\ell - 1)$ edges of this cycle, whence

$$e(C, B_2) = \sum_{x \in B_2} d_C(x) \leqslant \tfrac{1}{2}(\ell - 1)|B_2| + t\,,$$

where t denotes the number of triangles formed by a vertex in B_2 and an edge of the cycle. Further, by the second part of Claim 5.6, each edge of the cycle can sit in at most $\frac{1/15}{3r-2}n$ such triangles, wherefore $t \leqslant \frac{7/15}{3r-2}n$.

On the other hand, each $v \in C$ has at least $\frac{1}{2}|B_2| - \frac{1/15}{3r-2}n$ neighbours in B_2 due to $C \subseteq B_1 \setminus P$ and (5.4), whence

$$e(C, B_2) = \sum_{k=1}^{\ell} d_2(v_k) \geqslant \tfrac{1}{2}\ell|B_2| - \tfrac{7/15}{3r-2}n\,.$$

By combining all these estimates we infer

$$|B_2| \leqslant \tfrac{28/15}{3r-2}n\,,$$

which, however, violates (α). This concludes the proof of Fact 5.7.2. $\square$

Now consider the partition

$$B_1 \setminus P = Q \mathbin{\dot\cup} R \mathbin{\dot\cup} S$$

defined by

$$\begin{aligned} Q &= \Big\{x \in B_1 \setminus P \colon d_2(x) \leqslant \tfrac{1}{2}(|B_2| + \delta n)\Big\}\,, \\ R &= \Big\{x \in B_1 \setminus P \colon \tfrac{1}{2}(|B_2| + \delta n) < d_2(x) \leqslant \tfrac{7/4}{3r-2}n\Big\}\,, \\ \text{and} \qquad S &= \Big\{x \in B_1 \setminus P \colon \tfrac{7/4}{3r-2}n < d_2(x)\Big\}\,. \end{aligned}$$

Fact 5.7.3. *There is no edge connecting a vertex in S with a vertex in B_1.*

Proof. By (ζ) and the definition of S the common neighbourhood of such an edge would intersect B_2 in at least

$$\tfrac{7/4}{3r-2}n + \tfrac{1/3}{3r-2}n - |B_2| \overset{(\alpha)}{\geqslant} \tfrac{1/15}{3r-2}n$$

vertices, contrary to Claim 5.6. □

Fact 5.7.4. *The set $R \cup S$ is independent.*

Proof. Assume that we have an edge uv both of whose endvertices are in $R \cup S$. According to the definitions of R and S, the common neighbourhood J of u and v has at least δn vertices in B_2 and by $\alpha(G) < \delta n$ there exists an edge xy in $B_2 \cap J$.

We will now try to construct a K_{2r-4} in the common neighbourhood $J_* \subseteq J$ of u, v, x, and y, which would give a contradiction to $K_{2r} \nsubseteq G$. To this end we utilise Claim 5.3 with $I = [r] \smallsetminus \{1, 2\}$ and it remains to show that we have $|B_j \cap J_*| \geqslant \tfrac{1/15}{3r-2}n$ for every $j \in [3, r]$.

Thanks to Claim 5.6 we already know that x and y have at least $\tfrac{4/15}{3r-2}n$ common neighbours in each B_j with $j \in [3, r]$, so it suffices to prove $|B_j \cap J| \geqslant |B_j| - \tfrac{1/5}{3r-2}n$ instead. For this purpose it is enough to establish

$$|J \smallsetminus (B_1 \cup B_2)| \geqslant n - (|B_1| + |B_2|) - \tfrac{1/5}{3r-2}n\,. \tag{$\star$}$$

Now due to $u, v \in B_1 \smallsetminus P$ and (5.4) we have

$$|J \smallsetminus B_1| \geqslant 2\big(n - |B_1| - \tfrac{1}{2}|B_2| - \tfrac{1/15}{3r-2}n\big) - (n - |B_1|) = n - |B_1| - |B_2| - \tfrac{2/15}{3r-2}n$$

and Claim 5.6 tells us that

$$|J \cap B_2| \leqslant \tfrac{1/15}{3r-2}n\,.$$

It is easily seen that the last two estimates imply (⋆). □

We will now work towards an upper bound on $e(B_1 \smallsetminus P, B_2)$. Due to the definitions of Q, R, and S we have

$$\begin{aligned} e(B_1 \smallsetminus P, B_2) &\leqslant |Q| \cdot \tfrac{1}{2}(|B_2| + \delta n) + |R| \cdot \tfrac{7/4}{3r-2}n + |S||B_2| \\ &\overset{(\alpha)}{\leqslant} (|Q| + |R|) \cdot \tfrac{1}{2}(|B_2| + \delta n) + |R| \cdot \tfrac{4/5}{3r-2}n + |S||B_2|\,. \end{aligned}$$

According to Fact 5.7.4 and $\alpha(G) < \delta n$ we have $|R| \leqslant \delta n - |S|$ and thus we arrive at

$$\begin{aligned} e(B_1 \smallsetminus P, B_2) &\leqslant \tfrac{1}{2}|B_1 \smallsetminus P|(|B_2| + \delta n) + (\delta n - |S|)\tfrac{4/5}{3r-2}n + \tfrac{1}{2}|S|(|B_2| - \delta n) \\ &= \tfrac{1}{2}|B_1 \smallsetminus P||B_2| + \tfrac{1}{2}\delta n(|B_1 \smallsetminus P| + |B_2|) - \tfrac{1}{2}\delta^2 n^2 \\ &\qquad + (\delta n - |S|)\big(\tfrac{4/5}{3r-2}n + \tfrac{1}{2}\delta n - \tfrac{1}{2}|B_2|\big)\,. \end{aligned}$$

Employing (α) we may weaken this to

$$e(B_1 \smallsetminus P, B_2) \leqslant \tfrac{1}{2}|B_1 \smallsetminus P||B_2| + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2 - 2\delta n(\delta n - |S|)\,. \tag{5.6}$$

Next we learn from Lemma 5.1 and Fact 5.7.2 that $e(Q \cup R) \leqslant \alpha(Q \cup R)^2$, where $\alpha(Q \cup R)$, the size of the largest independent set in $Q \cup R$, is at most $\delta n - |S|$ due to Fact 5.7.3 and $\alpha(G) < \delta n$. So in other words we have $e(Q \cup R) \leqslant (\delta n - |S|)^2 \leqslant \delta n(\delta n - |S|)$. A further application of Fact 5.7.3 leads to the seemingly stronger inequality $e(B_1 \smallsetminus P) \leqslant \delta n(\delta n - |S|)$ and together with (5.6) this yields

$$e(B_1 \smallsetminus P, B_1 \cup B_2 \smallsetminus P) \leqslant \tfrac{1}{2}|B_1 \smallsetminus P||B_2| + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2 \,.$$

Adding the trivial upper bound for $e\big(B_1 \smallsetminus P, V \smallsetminus (B_1 \cup B_2)\big)$ we obtain

$$e(B_1 \smallsetminus P, V \smallsetminus P) \leqslant |B_1 \smallsetminus P|\big(n - |B_1| - \tfrac{1}{2}|B_2|\big) + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2 \,.$$

Combined with (5.5) this shows the desired estimate

$$e(B_1, V) \leqslant |B_1|\big(n - |B_1| - \tfrac{1}{2}|B_2|\big) + \tfrac{1}{2}\delta n(|B_1| + |B_2|) - \tfrac{1}{2}\delta^2 n^2$$

and the proof of Claim 5.7 is thereby complete. □

Finally, the addition of the r inequalities provided by the Claims 5.5 and 5.7 reveals

$$2e(G) = \sum_{i=1}^{r} e(B_i, V) \leqslant 2 \sum_{1 \leqslant i < j \leqslant r} |B_i||B_j| - |B_1||B_2| + \delta n \sum_{i=1}^{r} |B_i| - \delta^2 n^2$$

and Lemma 4.1 leads to

$$2e(G) \leqslant \big(\tfrac{3r-5}{3r-2} + \delta - \delta^2\big) n^2 \,.$$

Thereby Proposition 5.2 is proved. □

§6. THE PROOF OF THEOREM 1.2

Now the following should be clear.

Proposition 6.1. *For every integer $r \geqslant 2$ there exist an integer n_0 and a positive real number δ_0 such that for every $\delta \leqslant \delta_0$ every graph G on $n \geqslant n_0$ vertices with $K_{2r} \nsubseteq G$, $\delta(G) \geqslant \frac{3r-5}{3r-2}n$, and $\alpha(G) < \delta n$ has at most $\big(\frac{3r-5}{3r-2} + \delta - \delta^2\big)\frac{n^2}{2}$ edges.*

Proof. Let $\varepsilon > 0$ be the number provided by Proposition 5.2. By plugging it into Proposition 4.4 we obtain some constants $n_0 \in \mathbb{N}$ and $\delta_0 > 0$. Without loss of generality we may suppose that $\delta_0 \leqslant \varepsilon$. To check that these two numbers have the desired property we consider any graph G on $n \geqslant n_0$ vertices satisfying the above conditions for some $\delta \leqslant \delta_0 \leqslant \varepsilon$.

Now Proposition 4.4 informs us that G has an (r, ε)-exact partition and Proposition 5.2 yields the desired upper bound on $e(G)$. □

The only things which are currently missing from a proof of Theorem 1.2 are that we still need to abolish the minimum degree condition and n_0. This will be accomplished by means of a vertex deletion argument and an adjustment of δ_0.

Proof of Theorem 1.2. Let $n_0 \in \mathbb{N}$ and $\delta_0 \in (0,1)$ be as obtained by Proposition 6.1 and set

$$\delta_* = \tfrac{1}{r^2} \min\big(\delta_0^2, n_0^{-2}\big)\,. \tag{6.1}$$

Consider any n-vertex graph G satisfying

$$\alpha(G) < \delta n \quad \text{ and } \quad e(G) > \big(\tfrac{3r-5}{3r-2} + \delta - \delta^2\big)\tfrac{n^2}{2} \tag{6.2}$$

for some $\delta \leqslant \delta_*$. We are to prove that G contains a K_{2r} and our strategy for doing so is to find a subgraph $G' \subseteq G$ which otherwise would contradict Proposition 6.1. Define

$$q = e(G) - \tfrac{3r-2}{3r-5} \cdot \tfrac{n^2+n}{2}$$

and notice that $\delta n > \alpha(G) \geqslant 1$ entails

$$2q > (\delta - \delta^2)n^2 - \tfrac{3r-5}{3r-2}n > \big(1 - \delta - \tfrac{3r-5}{3r-2}\big)\delta n^2 \overset{(6.1)}{>} \big(\tfrac{1}{r} - \tfrac{1}{r^2}\big)\delta n^2 \geqslant \delta n^2/r^2\,.$$

We call a set of vertices $X \subseteq V(G)$ *solid* if

$$e(X) \geqslant \tfrac{3r-5}{3r-2} \cdot \tfrac{|X|^2+|X|}{2} + q\,.$$

Owing to the definition of q the whole vertex set $V(G)$ is solid and, consequently, there exists a maximal solid set $X' \subseteq V(G)$. Let G' denote the subgraph of G induced by X'. Utilising the trivial upper bound on the number of its edges we infer

$$|X'|^2 \geqslant 2e(G') \geqslant 2q \geqslant \delta n^2/r^2\,,$$

whence

$$r\sqrt{\delta} \cdot |X'| \geqslant \delta n > 1\,. \tag{6.3}$$

On the other hand, (6.1) implies $r\sqrt{\delta} \leqslant r\sqrt{\delta_*} \leqslant n_0^{-1}$ and thus we arrive at $|X'| \geqslant n_0$.

The maximality of X' tells us that for an arbitrary vertex $x \in X'$ the set $X' \smallsetminus \{x\}$ fails to be solid, meaning that

$$\tfrac{3r-5}{3r-2} \cdot \tfrac{|X'|^2+|X'|}{2} + q \leqslant e(X') = e(X' \smallsetminus \{x\}) + d(x) < \tfrac{3r-5}{3r-2} \cdot \tfrac{|X'|^2-|X'|}{2} + q + d(x)\,,$$

i.e., $d(x) > \frac{3r-5}{3r-2}|X'|$. Thereby we have proved that

$$\delta(G') > \tfrac{3r-5}{3r-2}|X'|\,.$$

The number $\delta' = \delta n/|X'|$ satisfies $\alpha(G') \leqslant \alpha(G) < \delta n = \delta'|X'|$ and due to (6.3) and (6.1) we have $\delta' \leqslant r\sqrt{\delta} \leqslant r\sqrt{\delta_*} \leqslant \delta_0$. So with the possible exception of $K_{2r} \nsubseteq G'$ all the assumptions of Proposition 6.1 hold for G' and δ'.

On the other hand, the solidity of X' yields

$$\begin{aligned}
2e(G') &\geqslant \tfrac{3r-5}{3r-2}|X'|^2 - \tfrac{3r-5}{3r-2}(n-|X'|) + \big(2e(G) - \tfrac{3r-5}{3r-2}n^2\big) \\
&\overset{(6.2)}{>} \tfrac{3r-5}{3r-2}|X'|^2 - \delta n(n-|X'|) + (\delta n^2 - \delta^2 n^2) \\
&= \big(\tfrac{3r-5}{3r-2} + \delta' - \delta'^2\big)|X'|^2\,,
\end{aligned}$$

meaning that the conclusion of Proposition 6.1 fails for G' and δ'. Therefore we have indeed that $K_{2r} \subseteq G' \subseteq G$. □

§7. ODD CLIQUES REVISITED

The goal of this section is to sketch a proof of the upper bound on $f_{2r+1}(\delta)$ provided by Theorem 1.3. As the case $r = 1$ is trivial, we concentrate on the case $r \geqslant 2$ in the sequel.

As a matter of fact, one can prove $f_{2r+1}(\delta) \leqslant \frac{r-1}{r} + \delta$ along the lines suggested by Section 3–6, but the details are much simpler because the first two classes of our partitions are not playing a special rôle anymore.

The required knowledge on coloured graphs concerns the collection

$$\mathscr{F}_{2r+1} = \{G_{2r+1,1}, \dots, G_{2r+1,r}\}\,.$$

The corresponding analogue of Proposition 3.4, likewise proved in [10], reads as follows.

Proposition 7.1. *Suppose that $r \geqslant 2$ and that G is a $\mathscr{F}_{2r+1}$-free coloured graph on n vertices with $\delta(G) > \frac{6r-8}{3r-1}n$. Then there is a partition $V(G) = W_1 \charfusion[\mathbin]{\cup}{\cdot} \dots \charfusion[\mathbin]{\cup}{\cdot} W_r$ such that all edges within the partition classes are green.*

By applying this to the reduced graph, the regularity lemma allows us to show the following.

Proposition 7.2. *Given an integer $r \geqslant 2$ and a real $\eta > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for every K_{2r+1}-free graph G on $n \geqslant n_0$ vertices with $\alpha(G) < \delta n$ and $\delta(G) \geqslant \frac{r-1}{r}n$ there is a partition*

$$V(G) = A_1 \,\dot\cup\, A_2 \,\dot\cup\, \dots \,\dot\cup\, A_r$$

with $e(A_i) \leqslant \eta n^2$ for all $i \in [r]$.

Due to the lower bound $e(G) \geqslant \frac{r-1}{r} \cdot \frac{n^2}{2}$ entailed by the minimum degree condition, such a partition needs to have the further properties

- $\big||A_i| - \frac{n}{r}\big| \leqslant \sqrt{2r\eta} \cdot n$ for every $i \in [r]$;
- and $e(A_i, A_j) \geqslant |A_i||A_j| - r\eta n^2$ whenever $i, j \in [r]$ are distinct.

We may now imitate the proof of Proposition 4.4 and start moving vertices around. This time, however, we just need to track the simpler quantity $\Omega_s = \sum_{i=1}^{r} e(A_i^s)$ and by making sure that it decreases by at least $\frac{n}{4r}$ in every step we arrive at the following.

Proposition 7.3. *For every $r \geqslant 2$ and $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that every K_{2r+1}-free graph G on $n \geqslant n_0$ vertices, with $\delta(G) \geqslant \frac{r-1}{r}n$ and $\alpha(G) < \delta n$ admits a partition*

$$V(G) = B_1 \cupdot \ldots \cupdot B_r$$

such that

- *$e(B_i) \leqslant \varepsilon n^2$ for $i = 1, \ldots, r$;*
- *$\left||B_i| - \frac{n}{r}\right| \leqslant \varepsilon n$ for $i = 1, \ldots, r$;*
- *and if $i, j \in [r]$ are distinct, then every vertex in B_i, has at least $\frac{n}{3r}$ neighbours in B_j.*

These properties are enough to make the idea from the proof of Claim 5.5 work and thus one can show $e(B_i, V) \leqslant (n - |B_i|)|B_i| + \delta n|B_i|$ for every $i \in [r]$. By adding these inequalities one obtains $e(G) \leqslant \left(\frac{r-1}{r} + \delta\right)\frac{n^2}{2}$.

To summarise, we have the following analogue of Proposition 6.1.

Proposition 7.4. *For every integer $r \geqslant 2$ there exist an integer n_0 and a positive real number δ_0 such that for every $\delta \leqslant \delta_0$ every graph G on $n \geqslant n_0$ vertices with $K_{2r+1} \nsubseteq G$, $\delta(G) \geqslant \frac{3r-5}{3r-5}n$, and $\alpha(G) < \delta n$ has at most $\left(\frac{r-1}{r} + \delta\right)\frac{n^2}{2}$ edges.*

Finally, the arguments from Section 6 allow us to remove the restriction on the minimum degree and to dispose of the additional parameter n_0. So altogether we have indeed $f_{2r+1}(\delta) \leqslant \frac{r-1}{r} + \delta$.

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