

# WEIGHTED VARIANTS OF THE ANDRÁSFAI-ERDŐS-SÓS THEOREM

CLARA MARIE LÜDERS AND CHRISTIAN REIHER

ABSTRACT. A well known result due to Andrásfai, Erdős, and Sós asserts that for  $r \geq 2$  every  $K_{r+1}$ -free graph on  $n$  vertices with  $\delta(G) > \frac{3r-4}{3r-1}n$  is  $r$ -partite. We study related questions in the context of weighted graphs, which are motivated by recent work on the Ramsey-Turán problem for cliques.

## §1. INTRODUCTION

1.1. **Simple graphs.** Extremal graph theory began with Turán’s discovery [20] that for  $r \geq 2$  every  $n$ -vertex graph  $G$  with more than  $\frac{r-1}{r} \cdot \frac{n^2}{2}$  edges contains a  $K_{r+1}$ , i.e., a clique on  $r+1$  vertices. The constant  $\frac{r-1}{r}$  appearing here is optimal, as can be seen by looking at balanced complete  $r$ -partite graphs. Simonovits [16] proved that this extremal configuration is subject to a stability phenomenon roughly saying that a  $K_{r+1}$ -free graph with almost the maximum number of edges is “almost”  $r$ -partite.

**Theorem 1.1** (Simonovits). *For every  $r \geq 2$  and  $\varepsilon > 0$  there exists some  $\delta > 0$  such that every  $K_{r+1}$ -free graph  $G$  on  $n$  vertices with at least  $(\frac{r-1}{r} - \delta) \frac{n^2}{2}$  edges admits a partition  $V(G) = W_1 \cup \dots \cup W_r$  satisfying  $\sum_{i=1}^r e(W_i) < \varepsilon n^2$ .*

A standard proof of Theorem 1.1 starts with the observation that an iterative deletion of vertices with small degree allows us to reduce to the case that  $\delta(G) > (\frac{r-1}{r} - \eta)n$  holds for an arbitrary constant  $\eta \ll \varepsilon$  chosen in advance. One then takes a clique of order  $r$  in  $G$ , whose existence is guaranteed by Turán’s theorem, and observes that the joint neighbourhoods  $\widetilde{W}_1, \dots, \widetilde{W}_r$  of the  $(r-1)$ -subsets of this clique are mutually disjoint independent sets, since otherwise  $G$  would contain a  $K_{r+1}$ . Finally, the minimum degree condition ensures that these sets cover all but at most  $r\eta n$  vertices of  $G$ , for which reason any partition  $V(G) = W_1 \cup \dots \cup W_r$  with  $W_i \supseteq \widetilde{W}_i$  for all  $i \in [r]$  has the desired property.

As a matter of fact, however, the second part of the argument can be omitted by appealing instead to a result of Andrásfai, Erdős, and Sós [1] telling us that an appropriate

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minimum degree condition of the form  $\delta(G) > \left(\frac{r-1}{r} - \eta\right)n$  implies that  $K_{r+1}$ -free graphs are  $r$ -partite. For an elegant alternative proof of this fact we refer to Brandt [4].

**Theorem 1.2** (Andrásfai, Erdős, and Sós). *Let  $G$  be for some  $r \geq 2$  a  $K_{r+1}$ -free graph on  $n$  vertices satisfying  $\delta(G) > \frac{3r-4}{3r-1}n$ . Then there is a homomorphism from  $G$  to  $K_r$ , i.e.,  $G$  is  $r$ -colourable.*

The constant  $\frac{3r-4}{3r-1}n$  appearing here is optimal. This can be seen by constructing for some  $n$  divisible by  $(3r-1)$  a graph  $G$  on a set  $V$  of  $n$  vertices having a partition

$$V = A_1 \cup \dots \cup A_5 \cup B_1 \cup \dots \cup B_{r-2}$$

such that

$$|A_1| = \dots = |A_5| = \frac{n}{3r-1} \quad \text{and} \quad |B_1| = \dots = |B_{r-2}| = \frac{3n}{3r-1},$$

and whose set of edges is as follows:

- there are all edges from a vertex in  $A_i$  to a vertex in  $A_{i+1}$ , where the indices are taken modulo 5;
- all edges from  $A_i$  to  $B_j$ , where  $i \in [5]$ ,  $j \in [r-2]$ ;
- and all edges from  $B_j$  to  $B_{j'}$ , where  $j, j' \in [r-2]$  are distinct.

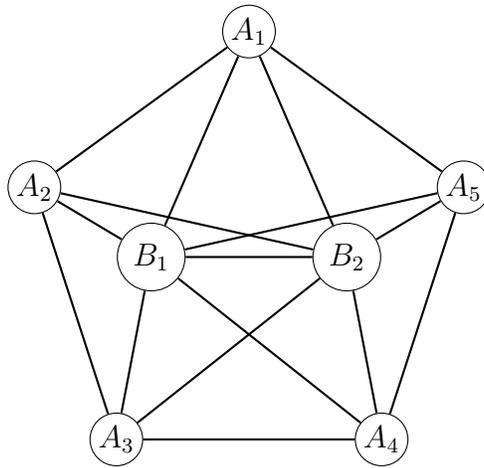


FIGURE 1.1. The extremal graph for the case  $r = 4$  of Theorem 1.2.

**1.2. Weighted graphs.** The concepts and problems discussed in the previous subsection make sense in the broader context of weighted graphs as well. For the purposes of this article, these are defined as follows.

**Definition 1.3.** A *weighted graph* is a pair  $G = (V, w)$  consisting of a finite *vertex set*  $V$  and a symmetric function  $w: V^2 \rightarrow \mathbb{R}_{\geq 0}$  such that  $w(x, x) = 0$  holds for all  $x \in V$ .

Here the word “symmetric” means that we require  $w(x, y) = w(y, x)$  for all  $x, y \in V$ . The notions of subgraphs, induced subgraphs, and isomorphisms extend in the following way from ordinary graphs to weighted graphs.

**Definition 1.4.** Let  $G = (V, w)$  and  $G' = (V', w')$  be two weighted graphs. We say that  $G'$  is a *subgraph* of  $G$  if  $V' \subseteq V$  and, additionally,  $w'(x, y) \leq w(x, y)$  holds for all  $x, y \in V'$ . If this conditions holds with equality throughout we call  $G'$  an *induced subgraph* of  $G$ . Finally,  $G$  and  $G'$  are said to be *isomorphic* if there is a bijection  $\varphi: V \rightarrow V'$  satisfying  $w(x, y) = w'(\varphi(x), \varphi(y))$  for all  $x, y \in V$ .

For two weighted graphs  $F$  and  $G$  we say that  $G$  is *F-free* if  $G$  does not possess any subgraph isomorphic to  $F$ . More generally, if  $\mathcal{F}$  is a set of weighted graphs such that  $G$  is *F-free* for every  $F \in \mathcal{F}$ , then  $G$  is said to be  *$\mathcal{F}$ -free*. The natural analogue of the “number of edges” of a weighted graph  $G = (V, w)$  is, of course, the quantity

$$e(G) = \frac{1}{2} \sum_{(x,y) \in V^2} w(x, y).$$

Now for every such set  $\mathcal{F}$  of weighed graphs and every finite set  $D \subseteq \mathbb{R}_{\geq 0}$  one may look at the *extremal function*  $n \mapsto \text{ex}_D(n, \mathcal{F})$  sending every positive integer  $n$  to the maximum of  $e(G)$  as  $G$  varies over  $\mathcal{F}$ -free weighted graphs of order  $n$  whose weight function does only attain values in  $D$ . The natural generalisation of Turán’s problem to this context asks to determine these functions for all choices of  $\mathcal{F}$  and  $D$ , the classical case being  $D = \{0, 1\}$ .

Similar as in this case, the *generalised Turán densities*

$$\pi_D(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}_D(n, \mathcal{F})}{n^2/2}$$

are easily shown to exist. Questions concerning  $\text{ex}_D(n, \mathcal{F})$  and  $\pi_D(\mathcal{F})$  are often studied in the literature, both for their own sake (see e.g. [9, 15]) and due to their connection with other parts of extremal combinatorics.

For instance, De Caen and Füredi [5] realised that such results addressing the case where  $D = \{0, 1, 2, 3, 4\}$  and  $\mathcal{F}$  denotes the set of all corresponding weighted graphs  $F$  on four vertices with  $e(F) \geq 21$  can be applied to the asymptotic solution of the hypergraph Turán problem for the Fano plane. Their approach is occasionally called the *link multigraph method* and led to many further results on Turán’s hypergraph problem (see also [2, 10–12, 14]).

Historically much earlier, a Turán problem for weighted graphs appeared in the determination of the so-called *Ramsey-Turán density* of even cliques due to Erdős, Hajnal, Szemerédi, and Sós [6]. This result belongs to an area initiated by Vera T. Sós, which is called *Ramsey-Turán theory*. Given a graph  $F$ , a number  $n$  of vertices, and a real number  $m > 0$  she defined the *Ramsey-Turán number*  $\text{RT}(n, m, F)$  to be the maximum

number of edges that an  $F$ -free graph  $G$  on  $n$  vertices with  $\alpha(G) < m$  can have. The problem is especially interesting if  $F$  is a clique and it is customary in this setting to pass to the *Ramsey-Turán density function*  $f_t: (0, 1) \rightarrow \mathbb{R}$  defined by

$$f_t(\delta) = \lim_{n \rightarrow \infty} \frac{\text{RT}(n, \delta n, K_t)}{n^2/2}.$$

A further simplification can be obtained by restricting the attention to the *Ramsey-Turán densities*

$$\varrho(K_t) = \lim_{\delta \rightarrow 0} f_t(\delta).$$

Such quantities have been intensively studied in the literature (see e.g. [3, 6, 7, 18] for important milestones and [17] for a beautiful survey). Owing to all these efforts it is known that

$$\varrho(K_t) = \begin{cases} \frac{t-3}{t-1} & \text{if } t \geq 3 \text{ is odd,} \\ \frac{3t-10}{3t-4} & \text{if } t \geq 4 \text{ is even.} \end{cases} \quad (1.1)$$

The even case is much harder and in their solution Erdős, Hajnal, Szemerédi, and Sós applied a result on the Turán density of a certain  $\{0, 1, 2\}$ -valued collection  $\mathcal{F}_t$  of weighted graphs to a reduced graph obtained by means of Szemerédi's regularity lemma [19]. These specific families  $\mathcal{F}_t$  of weighted graphs are introduced in Definition 1.5 below.

A few years ago, Fox, Loh, and Zhao [8] proved  $f_4(\delta) = \frac{1}{4} + \Theta(\delta)$ . In an attempt to generalise some of their arguments to larger even cliques we realised that the values of the Ramsey-Turán density function  $f_t(\delta)$  can be determined explicitly for  $\delta \ll t^{-1}$ . Notably in [13] we proved that

$$f_t(\delta) = \begin{cases} \frac{t-3}{t-1} + \delta & \text{if } t \geq 3 \text{ is odd,} \\ \frac{3t-10}{3t-4} + \delta - \delta^2 & \text{if } t \geq 4 \text{ is even} \end{cases}$$

holds provided that  $\delta$  is sufficiently small in a sense depending on  $t$ . In order to obtain these precise formulae we needed a stability result in the spirit of Theorem 1.1 but for the collections  $\mathcal{F}_t$  of weighted graphs mentioned above (see e.g. [13, Proposition 3.5]). While working on this subject, we proved the corresponding analogues of the Andrásfai-Erdős-Sós theorem as well. Eventually it turned out that these stronger results were, strictly speaking, not needed for the purposes of [13] and, since this article is already quite lengthy without them, we decided to publish them separately. They form the main contribution of the present work.

**1.3. Results.** Throughout the rest of this article we only need to deal with weighted graphs  $G = (V, w)$  satisfying  $w[V^2] \subseteq \{0, 1, 2\}$ . We regard such structures as *coloured* complete graphs on  $V$  by drawing a *green*, *blue*, or *red* edge between any two distinct vertices

$x, y \in V$  depending on whether  $w(x, y)$  attains the value 0, 1, or 2. For a nonnegative integer  $n$  the red and blue  $n$ -vertex clique are denoted by  $RK_n$  and  $BK_n$ , respectively. Moreover, for  $n \geq 2$  we mean by  $RK_n^-$  the graph obtained from an  $RK_n$  by recolouring one of its edges blue.

**Definition 1.5.** For two integers  $a \geq b \geq 1$  the coloured graph  $G_{a+b,b}$  of order  $a$  consists of an  $RK_b$  and a  $BK_{a-b}$  that are connected to each other by blue edges. Moreover, for every positive integer  $t$  we write  $\mathcal{F}_t = \{G_{t,i} : 1 \leq i \leq \frac{t}{2}\}$ .

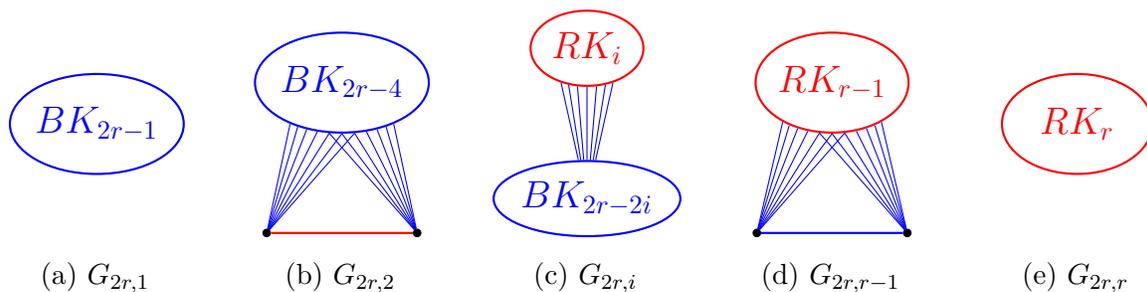


FIGURE 1.2. The family  $\mathcal{F}_{2r} = \{G_{2r,1}, G_{2r,2}, \dots, G_{2r,r}\}$ .

Erdős, Hajnal, Szemerédi, and Sós proved in [6] that

$$\pi_{\{0,1,2\}}(\mathcal{F}_t) = \begin{cases} \frac{2(t-3)}{t-1} & \text{if } t \geq 3 \text{ is odd,} \\ \frac{2(3t-10)}{3t-4} & \text{if } t \geq 4 \text{ is even,} \end{cases}$$

which in turn leads to (1.1) via the regularity method for graphs. In order to state the related results in the spirit of Theorem 1.2 one needs a notion of minimum degree for coloured graphs and it will be useful to have a notion of homomorphisms as well.

Now if  $G = (V, w)$  denotes a coloured graph and  $x \in V$ , it is natural to call

$$d(x) = \sum_{y \in V} w(x, y)$$

the *degree* of  $x$ . Moreover, the quantity  $\delta(G) = \min\{d(v) : v \in V\}$  will be referred to as the *minimum degree* of  $G$ .

**Definition 1.6.** A *homomorphism* from a weighted graph  $G = (V, w)$  to a weighted graph  $G' = (V', w')$  is a map  $\varphi : V \rightarrow V'$  with the property that any two distinct vertices  $x, y \in V$  satisfy  $w(x, y) \leq w'(\varphi(x), \varphi(y))$ .

For odd indices, we shall obtain the following.

**Theorem 1.7.** *Suppose that for some  $r \geq 2$  we have an  $\mathcal{F}_{2r+1}$ -free coloured graph  $G$  of order  $n$  with  $\delta(G) > \frac{6r-8}{3r-1}n$ . Then there is a homomorphism from  $G$  to  $RK_r$  or, explicitly, there is a partition*

$$V(G) = W_1 \cup \dots \cup W_r$$

*such that all edges within the partition classes are green.*

Consider the coloured graph obtained from the extremal graph described in Subsection 1.1 by replacing the edges there by red edges and colouring all other pairs green. This coloured graph has a minimum degree of exactly  $\frac{6r-8}{3r-1}n$  but, as it does not contain a  $BK_{r+1}$ , it cannot contain a member of  $\mathcal{F}_{2r+1}$  either. On the other hand, it does not admit a homomorphism to  $RK_r$  and thus it shows that the constant  $\frac{6r-8}{3r-1}$  appearing in Theorem 1.7 is optimal.

As in Ramsey Turán theory, the even case will be much harder.

**Theorem 1.8.** *Let  $r \geq 3$  be an integer and let  $G$  be a  $\mathcal{F}_{2r}$ -free coloured graph of order  $n$  with  $\delta(G) > \frac{14r-24}{7r-5}n$ . Then there is a homomorphism from  $G$  to  $RK_r^-$ . In other words, there is a partition*

$$V(G) = W_1 \cup \dots \cup W_r$$

*such that all edges within the partition classes are green and no edge from  $W_1$  to  $W_2$  is red.*

The reason why we stated this only for  $r \geq 3$  is that for  $r = 2$  one can easily show a stronger result. This is because  $\mathcal{F}_2$  consists only of a red edge and a blue triangle. Hence a direct application of the case  $r = 2$  of Theorem 1.2 shows that the desired conclusion can already be obtained from the weaker minimum degree assumption that  $\delta(G) > \frac{2}{5}n$ . For  $r \geq 3$ , however, the constant  $\frac{14r-24}{7r-5}$  appearing in Theorem 1.8 is optimal. This can be seen by taking  $n$  to be an arbitrary multiple of  $7r - 5$ , a vertex set  $V$  of size  $n$  with a partition

$$V = A_1 \cup \dots \cup A_{r-3} \cup B' \cup B'' \cup C' \cup C''$$

satisfying

$$|A_1| = \dots = |A_{r-3}| = \frac{7n}{7r-5}, \quad |B'| = |B''| = \frac{6n}{7r-5}, \quad \text{and} \quad |C'| = |C''| = \frac{2n}{7r-5},$$

and colouring

- the edges within the partition classes green,
- the edges from  $C'$  to  $C''$  green as well,
- the edges from  $B'$  to  $C'$  and from  $B''$  to  $C''$  blue,
- and all remaining edges red.



check that (2.1) is equivalent to

$$\sum_{z \in V(F)} \gamma_z \tilde{w}(x, z) \leq \left(2 - \frac{\delta(G)}{n}\right) \sum_{z \in V(F)} \gamma_z. \quad (2.2)$$

Our first argument of this form will establish the following lemma, which will later be used to show that a coloured graph  $G$  satisfying the assumption of either Theorem 1.7 or Theorem 1.8 cannot contain a  $BK_{r+1}$  (see Lemma 3.1 and Lemma 4.1 below).

**Lemma 2.1.** *Let  $q > b \geq 1$  be integers and suppose that  $G$  is a coloured graph on  $n$  vertices with  $\delta(G) > \left(2 - \frac{12}{3q+3b-5}\right)n$  containing a  $BK_q$ . Then either  $BK_{q+1}$  or  $G_{q+b,b}$  is a subgraph of  $G$ .*

*Proof.* Assume contrariwise that  $G$  is  $\{BK_{q+1}, G_{q+b,b}\}$ -free. For each integer  $k \in [0, b-1]$  we define

$$p_k = \max(0, k + q + 1 - 2b).$$

In view of

$$k + p_k = \max(k, 2k + q + 1 - 2b) \leq \max(b-1, q-1) < q \quad (2.3)$$

there exists a coloured graph  $H_k$  of order  $q$  having a vertex partition  $V(H_k) = A \cup B \cup C$  satisfying

- $|A| = k$ ,  $|B| = p_k$ ,  $|C| = q - (k + p_k)$ ,
- all edges of  $H_k$  connecting a vertex in  $A$  with a vertex in  $A \cup B$  are red,
- and all other edges of  $H_k$  are blue.

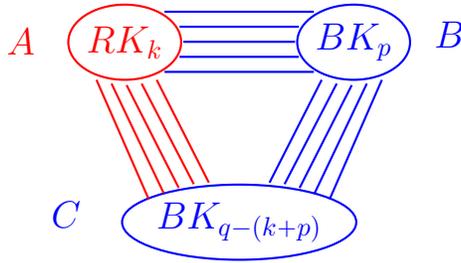


FIGURE 2.1. The coloured graph  $H_k$

Since all edges of  $H_0$  are blue and  $V(H_0)$  has size  $q$ , we know that  $G$  contains a copy of  $H_0$ . Now let  $k_*$  denote the largest integer in  $[0, b-1]$  with the property that  $G$  contains a copy of  $H_{k_*}$  and put  $p_* = p_{k_*}$ .

Let  $A \cup B \cup C \subseteq V(G)$  be the vertex set of such a copy with the notation as above. Notice that the calculation (2.3) shows  $C \neq \emptyset$ . So if  $k_* = b-1$ , then  $A$  and an arbitrary vertex in  $C$  would form an  $RK_b$ , while the remaining vertices in  $B \cup C$  would form a  $BK_{q-b}$ .

Due to the absence of green edges from  $A$  to  $B \cup C$  this means that  $G$  would contain a  $G_{q+b,b}$ , which is absurd.

This consideration proves

$$k_* \leq b - 2 \quad (2.4)$$

and our maximal choice of  $k_*$  entails that  $G$  does not contain an  $H_{k_*+1}$ .

*First Case:  $k_* < 2b - q - 1$ .*

This yields  $p_* = p_{k_*+1} = 0$  and  $B = \emptyset$ . We assign the weight 3 to the vertices in  $A$  and the weight 2 to the vertices in  $C$ . In view of  $q > b$  the total weight of all vertices in  $A \cup C$  is

$$3k_* + 2(q - k_*) = 2q + k_* \leq 2b + q - 2 + \frac{1}{2}(q - b - 1) = \frac{1}{2}(3q + 3b - 5).$$

Writing  $\gamma_z$  for the weight of every vertex  $z \in A \cup C$  we find, by the argument leading to (2.2), a vertex  $x \in V(G)$  satisfying

$$\sum_{z \in A \cup C} \gamma_z \tilde{w}(x, z) < \frac{6(3q + 3b - 5)}{3q + 3b - 5} = 6.$$

Owing to the integrality of the left side we obtain

$$3 \sum_{a \in A} \tilde{w}(x, a) + 2 \sum_{c \in C} \tilde{w}(x, c) \leq 5. \quad (2.5)$$

Because of  $BK_{q+1} \not\subseteq G$  there exists a vertex  $z \in A \cup C$  with  $\tilde{w}(x, z) = 2$ . In view of (2.5) this can only happen if  $z \in C$  and thus we infer

$$3 \sum_{a \in A} \tilde{w}(x, a) + 2 \sum_{c \in C \setminus \{z\}} \tilde{w}(x, c) \leq 1,$$

which in turn tells us that all members of  $A \cup C \setminus \{z\}$  are red neighbours of  $x$ . Consequently,  $(A \cup \{x\}) \cup (C \setminus \{z\})$  is the vertex partition of an  $H_{k_*+1}$  in  $G$  and we have reached a contradiction.

*Second Case:  $k_* \geq 2b - q - 1$ .*

Observe that now we have  $p_* = k_* + q + 1 - 2b$  and  $p_{k_*+1} = p_* + 1$ . This time we assign the weight  $\gamma_z = 2$  to every  $z \in A$  and the weight  $\gamma_z = 1$  to every  $z \in B \cup C$ . As before we find a vertex  $x \in V(G)$  with

$$\sum_{z \in A \cup B \cup C} \gamma_z \tilde{w}(x, z) < \frac{12(k_* + q)}{3q + 3b - 5} \stackrel{(2.4)}{<} \frac{12(q + b - 2)}{3(q + b - 2)} = 4,$$

i.e.,

$$2 \sum_{a \in A} \tilde{w}(x, a) + \sum_{y \in B \cup C} \tilde{w}(x, y) \leq 3.$$

Again the absence of a  $BK_{q+1}$  in  $G$  leads us to a vertex  $z \in B \cup C$  with  $\tilde{w}(x, z) = 2$ . Moreover, there are only red edges from  $x$  to  $A$  and at most one blue but no green edges from  $x$  to  $B \cup C \setminus \{z\}$ . This implies, however, that  $(A \cup \{x\}) \cup (B \cup C \setminus \{z\})$  supports an  $H_{k_*+1}$  in  $G$ , which is again a contradiction.  $\square$

### §3. THE PROOF OF THEOREM 1.7

An iterative application of Lemma 2.1 leads to the following result.

**Lemma 3.1.** *For  $r \geq 2$  every  $\mathcal{F}_{2r+1}$ -free coloured graph  $G$  of order  $n$  with  $\delta(G) > \frac{6r-8}{3r-1}n$  is  $BK_{r+1}$ -free.*

*Proof.* Let  $q$  be maximal with  $BK_q \subseteq G$  and assume for the sake of contradiction that  $q \geq r + 1$ . Due to  $G_{2r+1,1} = BK_{2r}$  we have  $q < 2r$  and, hence, the number  $b = 2r + 1 - q$  satisfies  $q > b \geq 1$ . Since  $\delta(G) > \left(2 - \frac{12}{3(2r+1)-5}\right)n$ , it follows from Lemma 2.1 that either  $BK_{q+1} \subseteq G$  or  $G_{2r+1,b} \subseteq G$ . The former, however, contradicts the maximality of  $q$  and the latter contradicts  $G$  being  $\mathcal{F}_{2r+1}$ -free.  $\square$

Now Theorem 1.7 follows by means of a simple application of the Andrásfai-Erdős-Sós theorem.

*Proof of Theorem 1.7.* Let  $H$  denote the simple graph on  $V(G)$  whose edges correspond to the blue or red edges of  $G$ . The minimum degree condition on  $G$  yields

$$\delta(H) \geq \frac{1}{2}\delta(G) > \frac{3r-4}{3r-1}n$$

and Lemma 3.1 tells us that  $H$  is  $K_{r+1}$ -free. So by Theorem 1.2  $H$  is  $r$ -partite and the claim follows.  $\square$

### §4. THE PROOF OF THEOREM 1.8

Again we begin by utilising Lemma 2.1.

**Lemma 4.1.** *For  $r \geq 3$  every  $\mathcal{F}_{2r}$ -free coloured graph  $G$  of order  $n$  with  $\delta(G) > \frac{14r-24}{7r-5}n$  is  $BK_{r+1}$ -free.*

*Proof.* As in the proof of Lemma 3.1 we look at the largest integer  $q$  with  $BK_q \subseteq G$  and observe that  $G_{2r,1} = BK_{2r-1}$  shows  $q \leq 2r - 1$ . So assuming  $q \geq r + 1$  Lemma 2.1 would again tell us that either  $BK_{q+1}$  or  $G_{2r,2r-q}$  is a subgraph of  $G$ , both of which is absurd. Actually, this argument only requires the lower bound  $\delta(G) > \left(2 - \frac{12}{6r-5}\right)n$  on the minimum degree of  $G$ , which is less than what we stated.  $\square$

In order to define the homomorphism demanded by Theorem 1.8 it would be tremendously helpful to know that no induced subgraph of  $G$  with three vertices has exactly one red edge. While not being true in general, this assertion will turn out to hold in the important special case that all edges of  $G$  that are not themselves red belong to the common red neighbourhood of some  $RK_{r-2}$  (see Lemma 4.7 below). This property of  $G$  can in turn be derived from a certain “edge-maximality” condition (see Lemma 4.3 and Lemma 4.6 below). The definition that follows facilitates talking about this plan.

**Definition 4.2.** Let  $G = (V, w)$  be a coloured graph.

- (a) If  $G$  is  $\mathcal{F}_{2r}$ -free and every  $\mathcal{F}_{2r}$ -free coloured graph  $G' = (V, w')$  having  $G$  as a subgraph (i.e., satisfying  $w'(x, y) \geq w(x, y)$  for all  $x, y \in V$ ) coincides with  $G$ , then we say that  $G$  is *extremal*.
- (b) A blue or green edge of  $G$  is called *secure* if it is contained in the common red neighbourhood of some  $RK_{r-2}$ .
- (c) A *wicked triangle* in  $G$  is a triple  $(x, y, z)$  of distinct vertices, such that  $xy$  is red and  $xz, yz$  are either blue or green. If in this situation both  $xz$  and  $yz$  are blue, then  $(x, y, z)$  is said to be a *blue wicked triangle*.

We shall see later that one only needs to deal with the extremal case when proving Theorem 1.8. As indicated above we will prove in this case that all blue and green edges are indeed secure and that wicked triangles do not exist. We commence with the easiest of these claims, the security of blue edges.

**Lemma 4.3.** *If  $G$  designates an extremal  $\mathcal{F}_{2r}$ -free coloured graph with  $n$  vertices and  $\delta(G) > \frac{14r-24}{7r-5}n$ , then all blue edges of  $G$  are secure.*

*Proof.* Let  $xy$  denote an arbitrary blue edge of  $G$ . By extremality, the weighted graph  $G'$  arising from  $G$  by recolouring  $xy$  red contains, for some  $i \in [r]$ , a subgraph isomorphic to  $G_{2r,i}$ . If  $i \neq r$  this subgraph would have at least  $r + 1$  vertices no two of which are connected by a green edge in  $G$ . Consequently,  $G$  would contain a  $BK_{r+1}$ , which contradicts Lemma 4.1. So  $G'$  contains an  $RK_r$  and, as  $G$  was  $RK_r$ -free, the vertices  $x$  and  $y$  must belong to this  $RK_r$ . Its remaining  $r - 2$  vertices form, in  $G$ , an  $RK_{r-2}$  whose red neighbourhood contains  $x$  and  $y$ .  $\square$

At this moment we could already rule out the existence of blue wicked triangles (see part (i) of Lemma 4.7 below). However, the argument for doing so is very similar to the proof that, provided the green edges are secure as well, there cannot be any wicked triangles at all. For this reason we postpone this step and consider the green edges first.

But it will be important to remember that we may already assume the absence of blue wicked triangles when treating the security of green edges.

As a further preparation towards this latter task we need to exclude a configuration that is closely tied to the example given at the end of the introduction demonstrating the optimality of the minimum degree condition in Theorem 1.8.

**Definition 4.4.** By  $J$  we mean the coloured graph of order  $r + 1$  with vertex set

$$A \cup \{b', b'', c', c''\},$$

where  $|A| = r - 3$ ,  $c'c''$  is green, and  $b'c', b''c''$  are blue, while all other edges are red.

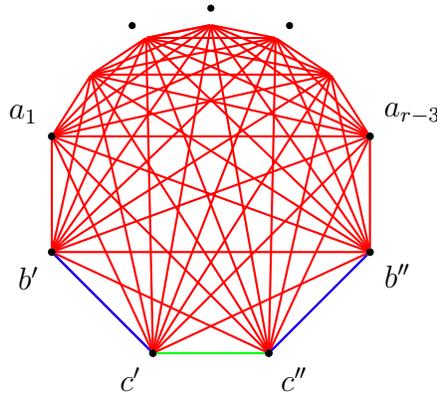


FIGURE 4.1. The coloured graph  $J$  with  $A = \{a_1, \dots, a_{r-3}\}$ .

**Lemma 4.5.** *A  $\{RK_r, BK_{r+1}\}$ -free coloured graph  $G$  of order  $n$  with  $\delta(G) > \frac{14r-24}{7r-5}n$  cannot contain  $J$  as subgraph.*

*Proof.* Otherwise let  $Q = A \cup \{b', b'', c', c''\} \subseteq V(G)$  be the vertex set of a copy of  $J$  in  $G$ , the notation being as in Definition 4.4. We assign weights to the members of  $Q$  according to the formula

$$\gamma_q = \begin{cases} 7 & \text{if } q \in A, \\ 6 & \text{if } q \in \{b', b''\}, \\ 2 & \text{if } q \in \{c', c''\}. \end{cases}$$

So the total weight of all vertices is  $7(r - 3) + 6 \cdot 2 + 2 \cdot 2 = 7r - 5$  and the standard argument leads to a vertex  $x \in V(G)$  with

$$\sum_{q \in Q} \gamma_q \tilde{w}(x, q) \leq 13. \quad (4.1)$$

This inequality allows us to analyse the set  $T = \{q \in Q: \tilde{w}(x, q) = 2\}$ . As an immediate consequence of (4.1) we have  $T \subseteq \{b', b'', c', c''\}$ . Moreover, the assumption  $b' \in T$  would

imply that  $Q \setminus \{b'\}$  is contained in the red neighbourhood of  $x$ , but then  $A \cup \{b'', c', x\}$  would induce an  $RK_r$  in  $G$ , which is absurd. By symmetry the same consideration applies to  $b''$  as well and thus we have  $T \subseteq \{c', c''\}$ .

Now it follows from  $A \cup \{b', b'', c', x\}$  not spanning a  $BK_{r+1}$  in  $G$  that  $c' \in T$  and, similarly, we get  $c'' \in T$  as well. By plugging  $T = \{c', c''\}$  into (4.1) we learn

$$6 \sum_{q \in A \cup \{b', b''\}} \tilde{w}(x, q) \leq 5,$$

and for this reason  $A \cup \{b', b''\}$  is part of the red neighbourhood of  $x$ . But this means that  $A \cup \{b', b'', x\}$  forms an  $RK_r$  in  $G$ , which is absurd.  $\square$

Now we proceed with the security of green edges. The argument starts similar to the proof of Lemma 4.3, but there will be more different cases to investigate.

**Lemma 4.6.** *Let  $G$  be an extremal  $\mathcal{F}_{2r}$ -free coloured graph with  $\delta(G) > \frac{14r-24}{7r-5}n$ . If  $G$  contains no blue wicked triangle, then all green edges of  $G$  are secure.*

*Proof.* Recall that  $G$  has to be  $\{BK_{r+1}, J\}$ -free by Lemma 4.1 and Lemma 4.5. Now consider any green edge  $xy$  of  $G$  and denote the coloured graph that one obtains from  $G$  when one recolours  $xy$  to become blue by  $G'$ . Due to the extremality of  $G$  we know that  $G'$  cannot be  $\mathcal{F}_{2r}$ -free. Exploiting that  $G$  is  $BK_{r+1}$ -free it is easily seen that  $G'$  must contain a  $G_{2r, r-1}$  with  $x$  and  $y$  among its vertices. This  $G_{2r, r-1}$  is, of course, only known to be a subgraph of  $G'$  that does not need to be induced. In fact, the absence of blue wicked triangles in  $G$  entail that “many” edges of this subgraph that “in general” would only be known to be either blue or red must actually be red. Taking into account that  $G$  is  $J$ -free a quick case analysis shows that one of the three situations displayed in Figure 4.2 arises.

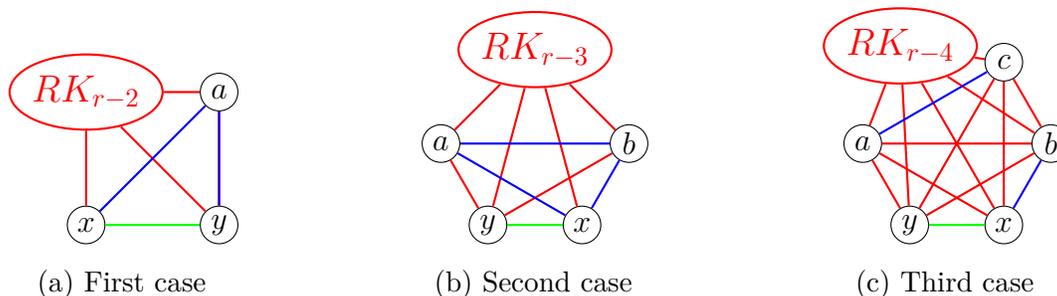


FIGURE 4.2. Possibilities for the edge  $xy$ .

Observe that it might still be the case that some of the edges drawn blue in these pictures are actually red in  $G$ . If the configuration depicted in Figure 4.2a occurs the edge  $xy$  is secure due to the  $RK_{r-2}$  shown there.

Suppose next that we are in the case shown in Figure 4.2b and let  $Q$  be the vertex set of the  $RK_{r-3}$ . Assign the weight 1 to  $a, b, x, y$ , and the weight 2 to the members of  $Q$ . So the total weight is  $2(r-1)$  and thus there is a vertex  $v$  with

$$\sum_{z \in \{a, b, x, y\}} \tilde{w}(v, z) + 2 \sum_{q \in Q} \tilde{w}(v, q) \leq 3.$$

In combination with neither  $Q \cup \{a, b, v, x\}$  nor  $Q \cup \{a, b, v, y\}$  forming a  $BK_{r+1}$  this implies that either  $\tilde{w}(a, v) = 2$  or  $\tilde{w}(b, v) = 2$ . By symmetry we may suppose that the latter holds, thus getting

$$\sum_{z \in \{a, x, y\}} \tilde{w}(v, z) + 2 \sum_{q \in Q} \tilde{w}(v, q) \leq 1.$$

It follows that all vertices in  $Q$  and at least two of  $a, x$ , and  $y$  are red neighbours of  $v$ . But if  $va$  and  $vy$  are red, then  $Q \cup \{a, v, y\}$  forms an  $RK_r$  in  $G$ , which is absurd. Furthermore, if  $va$  and  $vx$  are red, then  $Q \cup \{a, v, x, y\}$  forms a copy of  $J$  in  $G$ , which is not possible either. So the only remaining case is that  $vx$  and  $vy$  are red and then  $Q \cup \{v\}$  forms an  $RK_{r-2}$  exemplifying the security of  $xy$ .

It remains to discuss the configuration shown in Figure 4.2c, which can only arise if  $r \geq 4$ . This time we let  $Q$  denote the vertex set of the  $RK_{r-4}$ . Assigning the weight 4 to  $x, y$ , the weight 5 to  $a, b, c$ , and the weight 7 to the members of  $Q$  we have distributed a total weight of  $7r-5$  and in the usual manner we find a vertex  $v$  with

$$4 \sum_{z \in \{x, y\}} \tilde{w}(v, z) + 5 \sum_{z \in \{a, b, c\}} \tilde{w}(v, z) + 7 \sum_{q \in Q} \tilde{w}(v, q) \leq 13.$$

Exploiting that neither  $Q \cup \{a, b, c, x\}$  nor  $Q \cup \{a, b, c, y\}$  induces a  $BK_{r+1}$  we infer that  $\tilde{w}(v, \ell) = 2$  holds for some  $\ell \in \{a, b, c\}$ . Together with the above inequality this shows that all vertices in  $Q \cup \{a, b, c, x, y\}$  except for  $\ell$  are red neighbours of  $v$ . Due to the symmetry between  $a$  and  $c$  we may suppose that  $\ell \neq c$ . Now  $Q \cup \{a, v\}$  is the desired  $RK_{r-2}$  with  $xy$  in its neighbourhood.  $\square$

Finally, we deal with the alleged absence of wicked triangles.

**Lemma 4.7.** *Let  $G$  denote a  $\{RK_r, BK_{r+1}\}$ -free coloured graph of order  $n$  such that  $\delta(G) > \frac{14r-24}{7r-5}n$ .*

- (i) *If all blue edges of  $G$  are secure, then every wicked triangle of  $G$  possesses a green edge.*
- (ii) *If moreover the green edges of  $G$  are secure as well, then  $G$  contains no wicked triangles.*

*Proof.* Let  $V$  and  $w$  be the vertex set and weight function of  $G$ . Arguing indirectly we let  $(x, y, z)$  be a wicked triangle contradicting either of these two statements and such that subject to this  $w(x, z) + w(y, z)$  is as large as possible. Set  $\alpha = w(x, z)$  and  $\beta = w(y, z)$ . Notice that  $\alpha, \beta \in \{0, 1\}$  and  $xz, yz$  are secure. Consequently, there are two  $(r-2)$ -sets  $A, B \subseteq V$  inducing red cliques such that  $x, z$  belong to the common red neighbourhood of  $A$  while  $y, z$  belong to the common red neighbourhood of  $B$ . Let us select these sets  $A$  and  $B$  in such a way that  $k = |A \cap B|$  is maximal. Since  $(A \cap B) \cup \{x, y\}$  is a red clique and  $G$  is  $RK_r$ -free, we have

$$k \leq r - 3. \quad (4.2)$$

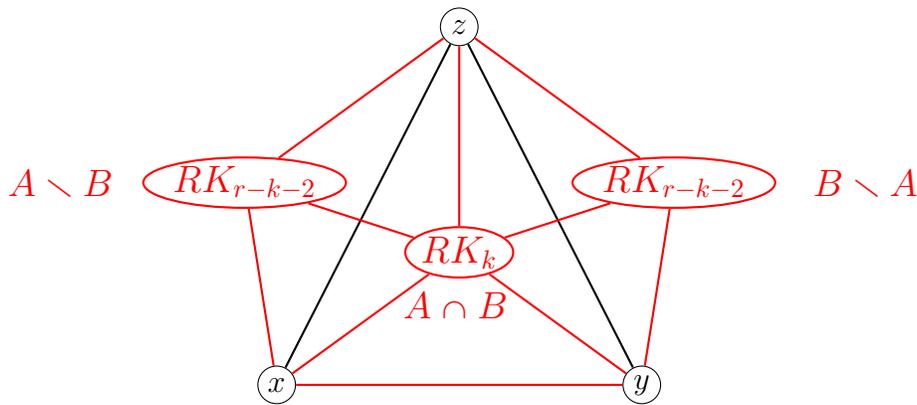


FIGURE 4.3. The sets  $A$  and  $B$ . The black pairs are either blue or green.

Notice that  $x, y, z \notin A \cup B$ , set  $Q = (A \cup B) \cup \{x, y, z\}$ , and assign weights to the vertices in  $Q$  by defining

$$\gamma_q = \begin{cases} 3 + \alpha - \beta & \text{if } q \in A \setminus B \text{ or } q = x, \\ 3 + \beta - \alpha & \text{if } q \in B \setminus A \text{ or } q = y, \\ 7 & \text{if } q \in A \cap B, \\ r - k + 1 & \text{if } q = z \end{cases}$$

for  $q \in Q$ . So the total weight of the vertices in  $Q$  is

$$6(r - k - 1) + 7k + (r - k + 1) = 7r - 5$$

and by our standard argument there exists a vertex  $v \in V$  with

$$\sum_{q \in Q} \gamma_q \tilde{w}(v, q) \leq 13. \quad (4.3)$$

It follows that

$$v \notin (A \cap B) \text{ and there is no green edge from } v \text{ to } A \cap B. \quad (4.4)$$

Put

$$a = \sum_{q \in A \cup \{x\}} \tilde{w}(v, q) \quad \text{and} \quad b = \sum_{q \in B \cup \{y\}} \tilde{w}(v, q)$$

and notice that (4.3) yields

$$(3 + \alpha - \beta)a + (3 + \beta - \alpha)b + 4\tilde{w}(v, z) \leq 13, \quad (4.5)$$

because (4.2) implies  $\gamma_z \geq 4$ .

Since  $A \cup \{x, v\}$  is not an  $RK_r$ , we have  $a \geq 1$  and, similarly,  $b \geq 1$ . So (4.5) yields that

$$vz \text{ is either blue or red.} \quad (4.6)$$

If  $\alpha = 1$ , i.e., if  $xz$  is blue, then the fact that  $A \cup \{v, x, z\}$  is not a  $BK_{r+1}$  entails  $a \geq 2$ . Performing the same argument for  $b$  we infer

$$a \geq 1 + \alpha \quad \text{and} \quad b \geq 1 + \beta. \quad (4.7)$$

Since  $\alpha^2 = \alpha$  and  $\beta^2 = \beta$ , we have

$$13 < 14 + (1 - \alpha)(1 - \beta) + \alpha\beta = (3 + \alpha - \beta)(4 - \alpha - \beta + \alpha\beta) + (3 + \beta - \alpha)(1 + \beta)$$

and by (4.5) and (4.7) this leads to

$$a \leq 3 - \alpha - \beta + \alpha\beta. \quad (4.8)$$

Now assume there would exist two distinct vertices in  $A \cup \{x\}$ , say  $s$  and  $t$ , that fail to be red neighbours of  $v$ . Then, in particular,  $v \notin \{s, t\}$  and the maximality of  $\alpha + \beta$  gives  $w(v, s) + w(v, t) \leq \alpha + \beta$ , whence

$$a \geq \tilde{w}(v, s) + \tilde{w}(v, t) \geq 4 - \alpha - \beta.$$

In view of (4.8) this is only possible if  $\alpha = \beta = 1$  and the foregoing estimate on  $a$  holds with equality. But then  $A \cup \{v, x, z\}$  is a  $BK_{r+1}$  in  $G$ , which is a contradiction. Therefore all but at most one vertex in  $A \cup \{x\}$  are red neighbours of  $v$ .

On the other hand,  $A \cup \{v, x\}$  is not a  $RK_r$  in  $G$ , so altogether we can conclude that there is a unique  $a^* \in A \cup \{x\}$  such that  $va^*$  is not red. Similarly, there is a unique  $b^* \in B$  such that  $vb^*$  is not red.

Next we suppose that  $vz$  would be blue. Then, in particular,  $v \notin (A \cup B)$  and the combination of (4.5) and (4.7) yields

$$13 \geq 4 + (1 + \alpha)(3 + \alpha - \beta) + (1 + \beta)(3 + \beta - \alpha) = 10 + 3(\alpha + \beta) + (\alpha - \beta)^2,$$

i.e.,  $\alpha = \beta = 0$ . So there is no wicked triangle with a blue edge and consequently there are only red edges from  $v$  to  $A \cup B$ . Thus  $a^* = x$  and  $b^* = y$ , for which reason  $(x, y, v)$  is a

wicked triangle. By the maximality of  $\alpha + \beta$  it follows that  $vx$  and  $vy$  are green, i.e., that  $a, b \geq 2$ . But now we get a contradiction to (4.5), which together with (4.6) proves that

$$vz \text{ is red.} \quad (4.9)$$

Since  $A \cup \{v, z\}$  cannot be an  $RK_r$ , it follows that  $a^* \in A$  and, similarly, we have  $b^* \in B$ . Owing to the uniqueness of  $a^*$  and  $b^*$  there are only two possibilities, namely  $a^* \in A \setminus B$  and  $b^* \in B \setminus A$ , or  $a^* = b^* \in A \cap B$ . If the former alternative would hold, then the sets  $A \cup \{v\} \setminus \{a^*\}$  and  $B \cup \{v\} \setminus \{b^*\}$  would contradict the maximality of  $k$ . So the only remaining case is that there is a member  $u = a^* = b^*$  of  $A \cap B$  such that  $Q \setminus \{u\}$  is in the red neighbourhood of  $v$ . By (4.4) the edge  $uv$  is blue. Since neither  $A \cup \{v, x, z\}$  nor  $B \cup \{v, y, z\}$  forms a  $BK_{r+1}$ , the edges  $xz$  and  $yz$  are green, i.e.,  $\alpha = \beta = 0$ . Let us recall that this means that there is no wicked triangle with a blue edge.

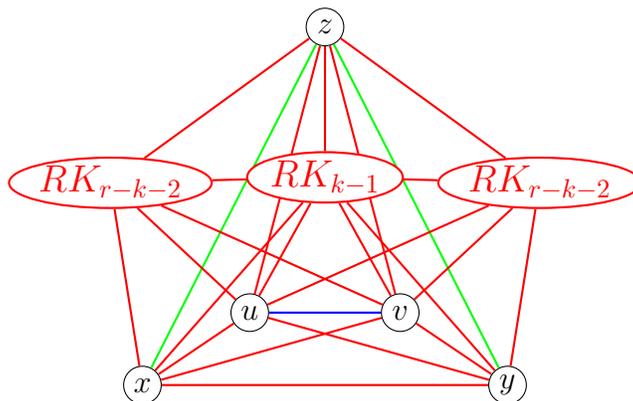


FIGURE 4.4. Current situation

At this moment the weights  $\gamma_q$  have done for us whatever they could do, and we proceed by assigning new weights to the vertices in  $Q$  and to  $v$ . To this end, we define

$$\eta_q = \begin{cases} 1 & \text{if } q \in (A \triangle B) \cup \{x, y, z\}, \\ 2 & \text{if } q \in \{u, v\}, \\ 3 & \text{if } q \in (A \cap B) \setminus \{u\} \end{cases}$$

for  $q \in Q \cup \{v\}$ . By (4.2) the total weight  $2r + k$  is at most  $3(r - 1)$  and thus there is a vertex  $t$  with

$$\sum_{q \in Q \cup \{v\}} \eta_q \tilde{w}(t, q) \leq 5. \quad (4.10)$$

We will now analyse the set  $T = \{q \in Q \cup \{v\} : \tilde{w}(q, t) = 2\}$ . By (4.10) it needs to be disjoint to  $A \cap B \setminus \{u\}$ . Suppose now that  $v \in T$ . Since the triangle  $(u, t, v)$  cannot be

wicked, it is not the case that  $ut$  is a red edge, which in turn yields  $\tilde{w}(u, t) + \tilde{w}(v, t) \geq 3$ , contrary to (4.10). This proves that  $v \notin T$  and by symmetry  $u \notin T$  holds as well.

Now it follows from  $G$  being  $BK_{r+1}$ -free that each of the four sets  $(A \setminus B) \cup \{x\}$ ,  $(A \setminus B) \cup \{z\}$ ,  $(B \setminus A) \cup \{y\}$ , and  $(B \setminus A) \cup \{z\}$  contains a member of  $T$ . On the other hand (4.10) yields  $|T| \leq 2$ . For these reasons, we have  $T = \{a^*, b^*\}$  for two vertices  $a^* \in A \setminus B$  and  $b^* \in B \setminus A$ .

Next we contend that  $S = (Q \cup \{v\}) \setminus T$  contains only red neighbours of  $t$ . To see this, consider an arbitrary  $s \in S$ . In view of  $s \notin T$  the edge  $st$  is either red or blue. Moreover, at least one of  $a^*$  or  $b^*$  is a red neighbour of  $s$ , so suppose that  $sa^*$  is red. Since  $(s, a^*, t)$  cannot be a wicked triangle with a blue edge, it follows that  $st$  is indeed red.

Now the sets  $A \cup \{t\} \setminus \{a^*\}$  and  $B \cup \{t\} \setminus \{b^*\}$  contradict the maximality of  $k$ .  $\square$

We conclude this section by giving the proof of our second main result.

*Proof of Theorem 1.8.* Let  $G' = (V, w')$  be an  $\mathcal{F}_{2r}$ -free coloured graph with the property that  $w'(x, y) \geq w(x, y)$  holds for all  $x, y \in V$  and such that subject to this condition  $e(G')$  is maximal. Then  $G'$  is extremal and satisfies  $\delta(G') \geq \delta(G) > \frac{14r-24}{7r-5}n$ . Since every homomorphism from  $G'$  to  $RK_r^-$  is also a homomorphism from  $G$  to  $RK_r^-$ , we may suppose for notational simplicity that  $G' = G$ , i.e., that  $G$  itself is extremal.

Now by Lemma 4.3 the blue edges of  $G$  are secure and Lemma 4.7 (i) informs us that  $G$  contains no blue wicked triangle. This in turn implies in view of Lemma 4.6 that the green edges of  $G$  are secure as well and, hence, Lemma 4.7 (ii) is applicable, showing that  $G$  contains no wicked triangles at all. This fact can be reformulated by saying that the reflexive and symmetric relation “ $w(x, y) \in \{0, 1\}$ ” is also transitive, i.e., an equivalence relation. Denote its (nonempty) equivalence classes by  $A_1, \dots, A_m$ . We will suppose moreover that this indexing has been arranged in such a way that for some integer  $s \in [0, m]$  each of the sets  $A_1, \dots, A_s$  spans at least one blue edge in  $G$ , whilst each of  $A_{s+1}, \dots, A_m$  forms a green clique.

For every  $i \in [m]$  we denote the minimum degree of the blue graph  $G$  induces on  $A_i$  by  $\alpha_i$ . Notice that

$$\frac{14r-24}{7r-5}n < \delta(G) \leq 2(n - |A_i|) + \alpha_i$$

holds for every  $i \in [m]$ , whence

$$2|A_i| - \alpha_i < \frac{14}{7r-5}n. \quad (4.11)$$

For  $i \in [s+1, m]$  we have  $\alpha_i = 0$  and the previous inequality simplifies to  $|A_i| < \frac{7}{7r-5}n$ . If, however,  $i \in [s]$ , then the trivial bound  $\alpha_i < |A_i|$  leads to  $|A_i| < \frac{14}{7r-5}n$ . By adding these

estimates up we obtain

$$n = \sum_{i=1}^m |A_i| < \frac{14s + 7(m-s)}{7r-5}n < \frac{m+s}{r-1}n,$$

wherefore  $m+s \geq r$ . On the other hand, by taking arbitrary blue edges from each of  $A_1, \dots, A_s$  as well as arbitrary vertices from each of  $A_{s+1}, \dots, A_m$  we can construct a  $BK_{m+s}$  in  $G$ . So in view of Lemma 4.1 we must have  $m+s = r$ . Similar arguments shows that the blue graphs induced by  $G$  on  $A_1, \dots, A_s$  are triangle-free. Moreover, one has  $s \geq 1$ , for otherwise  $G$  would contain an  $RK_r$ .

Now for each  $i \in [s]$  we find

$$\begin{aligned} n &= |A_i| + \sum_{j \neq i} |A_j| < |A_i| + \frac{14(s-1) + 7(m-s)}{7r-5}n \\ &= |A_i| + \frac{7(m+s-2)}{7r-5}n = |A_i| + \frac{7r-14}{7r-5}n \end{aligned}$$

and, consequently,  $|A_i| > \frac{9}{7r-5}n$ . In combination with (4.11) this leads to  $2|A_i| - \alpha_i < \frac{14}{9}|A_i|$ , i.e.,  $\alpha_i > \frac{4}{9}|A_i|$ . Since the blue graph  $G$  induces on  $A_i$  is triangle-free and  $\frac{4}{9} > \frac{2}{5}$ , the case  $r = 2$  of Theorem 1.2 entails that this blue graph is bipartite.

Thus for each  $i \in [s]$  there is a partition  $A_i = B_i \cup C_i$  such that  $B_i$  and  $C_i$  are green cliques in  $G$ . The structure we have thereby found in  $G$  may be regarded as a homomorphism from  $G$  to a coloured graph of order  $m+s = r$  having a blue matching of size  $s$  and otherwise red edges only. Due to  $s \geq 1$  this proves Theorem 1.8.  $\square$

## REFERENCES

- [1] B. Andrásfai, P. Erdős, and V. T. Sós, *On the connection between chromatic number, maximal clique and minimal degree of a graph*, Discrete Math. **8** (1974), 205–218. MR0340075 [↑1.1](#)
- [2] L. Bellmann and Chr. Reiher, *All Turán numbers of the Fano Plane*. Preprint. [↑1.2](#)
- [3] B. Bollobás and P. Erdős, *On a Ramsey-Turán type problem*, J. Combinatorial Theory Ser. B **21** (1976), no. 2, 166–168. MR0424613 [↑1.2](#)
- [4] S. Brandt, *On the structure of graphs with bounded clique number*, Combinatorica **23** (2003), no. 4, 693–696, DOI [10.1007/s00493-003-0042-z](#). MR2047472 [↑1.1](#)
- [5] D. De Caen and Z. Füredi, *The maximum size of 3-uniform hypergraphs not containing a Fano plane*, J. Combin. Theory Ser. B **78** (2000), no. 2, 274–276, DOI [10.1006/jctb.1999.1938](#). MR1750899 [↑1.2](#)
- [6] P. Erdős, A. Hajnal, V. T. Sós, and E. Szemerédi, *More results on Ramsey-Turán type problems*, Combinatorica **3** (1983), no. 1, 69–81, DOI [10.1007/BF02579342](#). MR716422 [↑1.2](#), [1.3](#)
- [7] P. Erdős and V. T. Sós, *Some remarks on Ramsey’s and Turán’s theorem*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 395–404. MR0299512 [↑1.2](#)
- [8] J. Fox, P.-S. Loh, and Y. Zhao, *The critical window for the classical Ramsey-Turán problem*, Combinatorica **35** (2015), no. 4, 435–476, DOI [10.1007/s00493-014-3025-3](#). MR3386053 [↑1.2](#)

- [9] Z. Füredi and A. Kündgen, *Turán problems for integer-weighted graphs*, J. Graph Theory **40** (2002), no. 4, 195–225, DOI [10.1002/jgt.10012](https://doi.org/10.1002/jgt.10012). MR1913847 [↑1.2](#)
- [10] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Probab. Comput. **14** (2005), no. 4, 467–484, DOI [10.1017/S0963548305006784](https://doi.org/10.1017/S0963548305006784). MR2160414 [↑1.2](#)
- [11] P. Keevash and D. Mubayi, *The Turán number of  $F_{3,3}$* , Combin. Probab. Comput. **21** (2012), no. 3, 451–456, DOI [10.1017/S0963548311000678](https://doi.org/10.1017/S0963548311000678). MR2912791 [↑1.2](#)
- [12] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica **25** (2005), no. 5, 561–574, DOI [10.1007/s00493-005-0034-2](https://doi.org/10.1007/s00493-005-0034-2). MR2176425 [↑1.2](#)
- [13] C. M. Lüders and Chr. Reiher, *The Ramsey-Turán problem for cliques*, available at [arXiv:1709.03352](https://arxiv.org/abs/1709.03352). Submitted. [↑1.2](#)
- [14] D. Mubayi and V. Rödl, *On the Turán number of triple systems*, J. Combin. Theory Ser. A **100** (2002), no. 1, 136–152, DOI [10.1006/jcta.2002.3284](https://doi.org/10.1006/jcta.2002.3284). MR1932073 [↑1.2](#)
- [15] V. Rödl and A. Sidorenko, *On the jumping constant conjecture for multigraphs*, J. Combin. Theory Ser. A **69** (1995), no. 2, 347–357, DOI [10.1016/0097-3165\(95\)90057-8](https://doi.org/10.1016/0097-3165(95)90057-8). MR1313901 [↑1.2](#)
- [16] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 279–319. MR0233735 [↑1.1](#)
- [17] M. Simonovits and V. T. Sós, *Ramsey-Turán theory*, Discrete Math. **229** (2001), no. 1-3, 293–340, DOI [10.1016/S0012-365X\(00\)00214-4](https://doi.org/10.1016/S0012-365X(00)00214-4). Combinatorics, graph theory, algorithms and applications. MR1815611 [↑1.2](#)
- [18] E. Szemerédi, *On graphs containing no complete subgraph with 4 vertices*, Mat. Lapok **23** (1972), 113–116 (1973) (Hungarian). MR0351897 [↑1.2](#)
- [19] ———, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401 (English, with French summary). MR540024 [↑1.2](#)
- [20] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452 (Hungarian, with German summary). MR0018405 [↑1.1](#)

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, HAMBURG, GERMANY

*E-mail address:* [Christian.Reiher@uni-hamburg.de](mailto:Christian.Reiher@uni-hamburg.de)

*E-mail address:* [Clara.Marie.Lueders@gmail.com](mailto:Clara.Marie.Lueders@gmail.com)