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Order types of models of reducts of Peano Arithmetic and their fragments

Abstract: It is well-known that non-standard models of Peano Arithmetic have order type $\mathbb{N} + \mathbb{Z} \cdot D$ where D is a dense linear order without first or last element. Not every order of the form $\mathbb{N} + \mathbb{Z} \cdot D$ is the order type of a model of Peano Arithmetic, though; in general, it is not known how to characterise those D for which this is the case. In this paper, we consider syntactic fragments of Peano Arithmetic (both with and without induction) and study the order types of their non-standard models. (Version 18; 12 July 2018)

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1 Introduction

1.1 Motivations & Results

The incompleteness phenomenon for arithmetic is due to the interaction of addition and multiplication: the theory of the natural numbers in the full language of arithmetic with addition and multiplication is essentially incomplete whereas its syntactic fragments in the language with only addition (known as *Presburger arithmetic*; cf. Presburger, 1930) and the language with only multiplication (known as *Skolem arithmetic*; cf. Skolem, 1930) are complete and decidable (Raatikainen, 2015, § 1.2.3). Addition and multiplication combined make theories *sequential*, i.e., they can encode the notion of finite sequence; this in turn paves the path to Gödel's incompleteness argument.

Non-standard models of arithmetic naturally split into archimedean classes (Definition 1.1) of elements with finite distance; a standard argument using only very basic properties of arithmetic shows that the order type of a non-standard model of arithmetic is of the form $\mathbb{N} + \mathbb{Z} \cdot D$ where D is a dense linear order

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without first or last element (cf. Kaye, 1991, Theorem 6.4). In general, it is not known which (uncountable) dense linear orders D give rise to an order type of a non-standard model of arithmetic (cf. Bovykin, 2000; Bovykin and Kaye, 2002, for an overview of what is known).

The three basic properties used in the standard argument mentioned in the last paragraph are (a) that the model is linearly ordered, (b) that addition is well-behaved with respect to that order, and (c) that every element is either even or odd. Given any standard axiomatization of PA, properties (a) and (b) do not need induction to be proved, while property (c) does. An inspection of the argument reveals that property (c) is crucial for both the arguments for density and the lack of a least element; so, we have linked induction to order-theoretic properties of the order D.

It is the aim of this paper to study the constraints on possible order types of non-standard models of fragments of PA. Usually, studies of fragments of PA focus on weakenings of the induction scheme to subclasses of formulae (cf., e.g., Carl *et al.*, 2017; Carl, 2016; Llewellyn-Jones, 2001; Zoethout, 2015); in this paper, we take a different approach and consider reducts of PA without and with the induction scheme.

We consider three operations, the unary successor operation and the binary addition and multiplication operations and their associated languages: $\mathcal{L}_{<,\mathrm{s}} := \{0, <, \mathrm{s}\}$, the language with an order relation and the successor operation, $\mathcal{L}_{<,\mathrm{s},+} := \{0, <, \mathrm{s}, +\}$, the language augmented with addition, and $\mathcal{L}_{<,\mathrm{s},+,\cdot} := \{0, <, \mathrm{s}, +, \cdot\}$, the full language of arithmetic. For each of the languages, we shall define the appropriate arithmetical axiom systems and the corresponding axiom schemes of induction, resulting a total of six theories,

$$\begin{array}{ccc} \mathsf{SA}^- & \subseteq & \mathsf{SA} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathsf{Pr}^- & \subseteq & \mathsf{Pr} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathsf{PA}^- & \subseteq & \mathsf{PA}, \end{array}$$

where the theories in the left column are without induction and the theories in the right column are with the axiom scheme of induction (for definitions, cf. § 1.2).

As usual, we use the following syntactic abbreviations: for $n \in \mathbb{N}$ and a variable x, we write

$$\mathrm{s}^n(x) := \underbrace{\mathrm{s}(\ldots(\mathrm{s}}_{n \text{ times.}}(x))\ldots) \text{ and}$$

$$nx := \underbrace{x + \ldots + x}_{n \text{ times.}}.$$

We shall show that SA^- proves the axiom scheme of induction for $\mathcal{L}_{<,\mathrm{s}}$ (Theorem 2.2) and hence SA^- and SA are the same theory, reducing our diagram to five theories. The main result of this paper is the separation of the remaining five theories in terms of order types: in the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S". In § 6, we shall show that the diagram is complete in the sense that if there is no arrow from T to S, then there is an order that is the order type of a model of T that cannot be the order type of a model of S.

$$\begin{tikzcd}
& \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA}. \arrow[u] \arrow[ll]
\end{tikzcd}$$

1.2 Definitions

In this section, we shall introduce the axiomatic systems whose order types we shall study in this paper. The axioms come in four groups corresponding to the order relation, the successor function, addition, and multiplication.

The order axioms O1 to O4 express that $<$ describes a linear order with least element 0 (O1 is trichotomy, O2 is transitivity, and O3 is irreflexivity):

$$x < y \lor x = y \lor x > y, \tag{O1}$$

$$(x < y \land y < z) \to x < z, \tag{O2}$$

$$\neg(x < x), \tag{O3}$$

$$x = 0 \lor 0 < x. \tag{O4}$$

The successor axioms S1 to S4 express that $<$ is discrete and that s is the successor operation with respect to $<$:

$$x = 0 \leftrightarrow \neg\exists y x = \mathrm{s}(y), \tag{S1}$$

$$x < y \to y = \mathrm{s}(x) \lor \mathrm{s}(x) < y, \tag{S2}$$

$$x < y \to \mathrm{s}(x) < \mathrm{s}(y), \tag{S3}$$

$$x < \mathrm{s}(x). \tag{S4}$$

Taken together, the axioms O1 to O4 and S1 to S4 (later called SA^-) constitute the theory of discrete linear orders with a minimum and a strictly increasing successor function.

The addition axioms P1 to P5 express the fact that $+$ and $<$ satisfy the axioms of ordered abelian monoids:

$$(x + y) + z = x + (y + z), \tag{P1}$$

$$x + y = y + x, \tag{P2}$$

$$x + 0 = x, \tag{P3}$$

$$x < y \to x + z < y + z, \tag{P4}$$

$$x + \mathrm{s}(y) = \mathrm{s}(x + y). \tag{P5}$$

Axiom ⋇ expresses the fact that if $x < y$, then the difference between them exists:

$$x < y \to \exists z x + z = y. \tag{⋇}$$

The multiplicative axioms M1 to M6 express that $\cdot$ and $+$ are commutative semi-ring operations respecting $<$:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{M1}$$

$$x \cdot y = y \cdot x, \tag{M2}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z, \tag{M3}$$

$$x \cdot \mathrm{s}(0) = x, \tag{M4}$$

$$x \cdot \mathrm{s}(y) = (x \cdot y) + x, \tag{M5}$$

$$x < y \wedge z \neq 0 \to x \cdot z < y \cdot z. \tag{M6}$$

Finally we have a schema of induction axioms.

$$(\varphi(0, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to (x + 1, \bar{y})) \to \forall x \varphi((x, \bar{y}). \tag{Ind_φ}$$

When considering subsystems of these axioms, we shall denote the axiom schema of induction restricted to the formulas of a language $\mathcal{L}$ by $\mathrm{Ind}(\mathcal{L})$. We shall consider the following systems of axioms:

$$
\begin{aligned}
\mathsf{SA}^- &= \mathrm{O1} + \mathrm{O2} + \mathrm{O3} + \mathrm{O4} + \mathrm{S1} + \mathrm{S2} + \mathrm{S3} + \mathrm{S4},\\
\mathsf{SA} &= \mathsf{SA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s}}),\\
\mathsf{Pr}^- &= \mathsf{SA}^- + \divideontimes + \mathrm{P1} + \mathrm{P2} + \mathrm{P3} + \mathrm{P4} + \mathrm{P5},\\
\mathsf{Pr} &= \mathsf{Pr}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+}),\\
\mathsf{PA}^- &= \mathsf{Pr}^- + \mathrm{M1} + \mathrm{M2} + \mathrm{M3} + \mathrm{M4} + \mathrm{M5} + \mathrm{M6},\\
\mathsf{PA} &= \mathsf{PA}^- + \mathrm{Ind}(\mathcal{L}_{<,\mathrm{s},+,\cdot});
\end{aligned}
$$

standing for 'Successor Arithmetic', 'Presburger Arithmetic', and 'Peano Arithmetic', respectively. Note that SA should not be confused with the theory $\mathrm{Th}(\mathbb{Q},+)$ called SA in (Hosono and Ikeda, 1994; Smoryński, 1991) (the 'S' there stands for 'Skolem').

In his original paper, Presburger (1930) uses a different axiomatisation of Presburger Arithmetic that we shall call Pr^{D}. The axioms of Pr^{D} are the axioms for discretely ordered abelian additive monoids with smallest non-zero element 1, axiom P4, and the following axiom schema:

$$\forall x \exists y x = ny \vee x = \mathrm{s}(ny) \vee \ldots \vee x = \mathrm{s}^{n-1}(ny), \tag{D_n}$$

for $0 < n \in \mathbb{N}$. (Note that D_2 is the statement "every number is either even or odd" called *property (c)* in our informal argument in § 1.1.) Presburger's famous theorem shows that Pr^{D} axiomatises the complete theory $\mathrm{Th}(\mathbb{N},+)$. Since our Pr clearly implies Pr^{D}, it also axiomatises $\mathrm{Th}(\mathbb{N},+)$.

In this paper we do not take into consideration Skolem arithmetic SK, i.e., the multiplicative fragment of PA. This is due to the fact that SK, usually defined as $\mathrm{Th}(\mathbb{N},\cdot)$, does not carry an order structure, i.e., the order is not definable in $\mathcal{L}_{\cdot}$. Moreover, adding the order to Skolem arithmetic makes it much more expressive: Robinson showed that addition is definable in $\mathrm{Th}(\mathbb{N},<,\cdot)$, and thus $\mathrm{Th}(\mathbb{N},<,\cdot)$ is essentially full arithmetic (Robinson, 1949). Therefore, an analysis of Skolem arithmetic in terms of order types is not fruitful.

1.3 Order types

As usual, order types are the isomorphism classes of partial orders. If $\mathcal{L}$ is any language containing $<$ and M is an $\mathcal{L}$-structure, by a slight abuse of language, we refer to the $\{<\}$-reduct of M as its *order type*. In situations where the order structure is clear from the context, we do not explicitly include it in the notation: e.g., the notation $\mathbb{Z}$ refers to both the set of integers and the ordered structure $(\mathbb{Z},<)$ with the natural order $<$ on $\mathbb{Z}$.

Let $(A, <)$ be a linearly ordered set and $(B, 0, <)$ be linearly ordered set with a least element 0. Given a function f from A to B, we shall call the set

$$\mathrm{supp}(f) = \{b \in B\,;\, b = 0 \vee f(b) \neq 0\}$$

the *support of* f. As usual, we say that a subset $S \subseteq A$ is *reverse well-founded* if it has no strictly increasing infinite sequences. Given a function $f : A \to B$ whose support is reverse well-founded, we call the maximum element of the support of f the *leading term of* f and denote it by $\textsc{lt}(f)$.

If A and B are two linear orders, then A^* is the inverse order of A, $A + B$ is the order sum, and $A \cdot B$ is the product order. Moreover, if A has a least element 0 then A^B is the set of functions with finite support from B to A ordered anti-lexicographically. Note that in the case that A and B are ordinal numbers, then the above operations correspond to the classical ordinal operations.

If $a \in A$, we denote the *initial segment defined by* a as $\mathrm{IS}(a) := \{b \in A\,;\, b < a\}$ and the *final segment defined by* a as $\mathrm{FS}(a) := \{b \in A\,;\, a < b\}$.

If $(G, 0, <, +)$ is an ordered abelian group (i.e., satisfies the axioms O1 to O4 and P1 to P4), then we define $G^+ := \{g \in G\,;\, 0 < g\} = \mathrm{FS}(0)$ to be the *positive part of* G. We call linear orders *groupable* if and only if there is an ordered abelian groupr with the same order type.

Let G be an ordered additive group. We define the *standard monoid over* G as the ordered monoid $(\mathbb{N} + \mathbb{Z} \cdot G^+, <, +)$ where $<$ is the order relation of $\mathbb{N} + \mathbb{Z} \cdot G^+$ and $+$ is defined point-wise, i.e.,

$$x + y = \begin{cases} n + m & \text{if } x = n,\ y = m \text{ and } m, n \in \mathbb{N}, \\ \langle z + x, g\rangle & \text{if } x \in \mathbb{N} \text{ and } y = \langle z, g\rangle \in \mathbb{Z} \cdot G^+, \\ \langle z + y, g\rangle & \text{if } y \in \mathbb{N} \text{ and } x = \langle z, g\rangle \in \mathbb{Z} \cdot G^+, \\ \langle z_x + z_y, g_x + g_y\rangle & \text{if } x = \langle z_x, g_x\rangle \in \mathbb{Z} \cdot G^+ \text{ and } y = \langle z_y, g_y\rangle \in \mathbb{Z} \cdot G^+. \end{cases}$$

It is easy to see that for each ordered group G the standard monoid over G is indeed a positive monoid.

If $(B, <, +)$ is any ordered group and X is a variable, we can consider the set $B[X]$ of polynomials in the variable X over B, consisting of terms $f = b_n X^n + \ldots + b_1 X + b_0$ where if $n \neq 0$ then $b_n \neq 0$, the *degree* of a polynomial is the highest occurring exponent, i.e., $\deg(f) = n$. We order polynomials as follows:

$$b_n X^n + \ldots + b_1 X + b_0 < c_m X^m + \ldots c_1 X + c_0$$

if either $n < m$ or $n = m$ and $b_i < c_i$ where i is the largest index such that $b_i \neq c_i$. This order respects addition and multiplication of polynomials in the sense of axioms P4 and M6, respectively. A polynomial is called *positive* if it is larger than the zero-polynomial in this order. If we define

$$\begin{aligned} O_0 &= \varnothing, \\ O_{\gamma+1} &= O_\gamma + \mathbb{Z}^\gamma \cdot \mathbb{N} \\ O_\lambda &= \bigcup_{\gamma\in\lambda} O_\gamma \text{ for } \lambda \text{ limit}, \end{aligned}$$

then for every natural number $n > 0$, the linear order O_n is the order type of non-negative polynomials with integer coefficients of degree at most $n-1$ and thus O_ω is the order type of all non-negative polynomials with integer coefficients.

1.4 Basic Properties

In this section, we shall remind the reader about basic tools of model theory of PA. We refer the reader to (Kaye, 1991) for a comprehensive introduction to the theory of non-standard models of PA. One of the main tools in studying the order types of models of PA is the concept of *archimedean class.*

Definition 1.1. Let M be a model of SA^-. Given $x, y \in M$ we say that *x and y are of the same magnitude*, in symbols $x \sim y$, if there are $m, n \in \mathbb{N}$ such that $\mathrm{s}^n(y) \geq x$ and $y \leq \mathrm{s}^m(x)$. The relation $\sim$ is an equivalence relation. For every $x \in M$, we shall denote by $[x]$ the equivalence class of x with respect to $\sim$ called the *archimedean class of x.*

The archimedean classes of a model of SA^- partition the model into convex blocks: if $y, w \in [x]$ and $y < z < w$, then $z \in [x]$ (the reader can check that only the axioms of SA^- are needed for this). Therefore, the quotient structure $M/{\sim}$ of archimedean classes is linearly ordered by the relation $<$ defined by $[x] < [y]$ if and only if $x < y$ and $[x] \neq [y]$. Furthermore, $[0]$ is the least element of the quotient structure. We refer to the classes that are different from $[0]$ as the *non-zero archimedean classes.* In particular, if A is the order type of the non-zero archimedean classes of M, then the order type of M is $\mathbb{N} + \mathbb{Z} \cdot A$.

So far, we worked entirely in the language $\mathcal{L}_{<,\mathrm{s}}$ with just the axioms of SA^-. If we also have addition in our language, we observe:

Lemma 1.2. *Let M be a non-standard model of* Pr^- *and $a \in M$ be a non-standard element of M. Then for every $n, m \in \mathbb{N}$ such that $n < m$ we have $[na] < [ma]$. In particular, if $\mathbb{N} + \mathbb{Z} \cdot A$ is the order type of M, then A does not have a largest element.*

Proof. Assume that $n < m$. We want to prove that $[na] < [ma]$. Let $n' > 0$ be such that $m = n + n'$. Let $i \in \mathbb{N}$ we want to show that $na + \mathrm{s}^i(0) < ma$. By definition $ma = (n + n')a = na + n'a$. Now by monotonicity of $+$ and by the fact that a is non-standard and $n' > 0$ we have $na + \mathrm{s}^i(0) < na + a = (n+1)a \leq (n+n')a = ma$. Therefore $[na] < [ma]$ as desired. □

Another important tool in the classical study of order types of models of PA is the *overspill* property:

Definition 1.3. Let M be a model of SA^-. Then $I \subseteq M$ is a *cut* of M if it is an initial segment of M with respect to $<$ and it is closed under s, i.e., for every $i \in I$ we have $\mathrm{s}(i) \in I$. A cut of M is *proper* if it is neither empty nor M itself.

Definition 1.4. Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language. A theory $T \supseteq \mathsf{SA}^-$ has the *$\mathcal{L}$-overspill property* if for every model $M \models T$ there are no $\mathcal{L}$-definable proper cuts of M.

Overspill is essentially a notational variant of induction:

Theorem 1.5. *Let $\mathcal{L} \supseteq \mathcal{L}_{<,\mathrm{s}}$ be a language and $T \supseteq \mathsf{SA}^-$ be any theory. Then the following are equivalent:*
(i) $\mathrm{Ind}(\mathcal{L}) \subseteq T$ *and*
(ii) T *has the $\mathcal{L}$-overspill property.*

Proof. "(i)⇒(ii)". Let $M \models T$ and I be a proper cut of M. Then $0 \in I$. Suppose towards a contradiction that I is definable by an $\mathcal{L}$-formula φ. Then Ind_φ implies that $I = M$, so I was not proper.

"(ii)⇒(i)". Assume that $\mathrm{Ind}_\varphi \notin T$ for some $\mathcal{L}$-formula φ and find $M \models T$ such that $M \models \neg\mathrm{Ind}_\varphi$. Define the formula $\varphi'(x) := \varphi(x) \wedge \forall y(y < x \rightarrow \varphi(y))$. Then φ' defines a proper cut in M, and thus, T does not have the $\mathcal{L}$-overspill property. □

In particular, SA, Pr, and PA have the overspill property for their respective languages $\mathcal{L}_{<,\mathrm{s}}$, $\mathcal{L}_{<,\mathrm{s},+}$, and $\mathcal{L}_{<,\mathrm{s},+,\cdot}$.

2 Successor Arithmetic

We begin our study by considering the two subsystems obtained by restricting our language to $\mathcal{L}_{<,\mathrm{s}}$, viz. SA^- and SA. The theory SA^- is the theory of discrete linear orders with a minimum and a strictly increasing successor function.

Lemma 2.1. *The theory* SA^- *satisfies quantifier elimination.*

Proof. It is enough to prove that for every quantifier free formula $\chi(\overline{x}, y)$ there is a quantifier free formula φ such that

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x})$$

where y does not appear in φ. We prove this claim by induction over χ. The only interesting cases are the atomic formulas.

If $\chi(x, y) \equiv \mathrm{s}^n(x) < \mathrm{s}^m(y)$: let $\varphi \equiv x = x$. Let $M \models \mathsf{SA}^-$, we want to show $M \models \exists y \chi(x, y)$. First assume $m \geq n$. Since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^{m+1}(x)$ we have $M \models \exists y \mathrm{s}^n(x) < \mathrm{s}^m(y)$ as desired. Otherwise if $n > m$ since $\mathsf{SA}^- \vdash \forall x x < \mathrm{s}^{(n-m)+1}(x)$ then $M \models \exists y \chi(\overline{x}, y)$. Hence:

$$\mathsf{SA}^- \models \exists y \chi(\overline{x}, y) \leftrightarrow \varphi(\overline{x})$$

as desired.

If $\chi(x, y) \equiv \mathrm{s}^n(y) < \mathrm{s}^m(x)$: first assume $m > n$ then since $\mathsf{SA}^- \vdash \forall x \mathrm{s}^n(x) < \mathrm{s}^m(x)$ we have $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow x = x$. If $m \leq n$ then $\mathsf{SA}^- \vdash \exists y \chi(x, y) \leftrightarrow \mathrm{s}^n(0) < \mathrm{s}^m(x)$. Indeed, let $M \models \mathsf{SA}^-$ be a model such that there is a $y \in M$ such that $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x)$ and $M \models \neg \mathrm{s}^n(0) < \mathrm{s}^m(x)$. We have two cases: if $M \models \mathrm{s}^n(0) = \mathrm{s}^m(x)$ then we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) = \mathrm{s}^n(0)$ but since $M \models \forall x \mathrm{s}^n(x) < \mathrm{s}^n(y) \to x < y$ then we would have $M \models y < 0$. If $M \models \mathrm{s}^m(x) < \mathrm{s}^n(0)$ again we would have $M \models \mathrm{s}^n(y) < \mathrm{s}^m(x) < \mathrm{s}^n(0)$ which implies $M \models y < 0$. On the other hand if $M \models \mathrm{s}^n(0) < \mathrm{s}^m(x)$ then trivially $M \models \exists y \chi(\overline{x}, y)$ as desired.

If $\chi(\overline{x}, y)$ does not have occurrences of y: then $\exists y \chi(\overline{x}, y)$ is either equivalent to $0 = 0$ or $\neg(0 = 0)$.

If $\chi(x, y) \equiv \mathrm{s}^n(x) = \mathrm{s}^m(y)$: similar to the second case. $\square$

Note that this proof is essentially in (Enderton, 2001, Theorem 32A) where Enderton shows quantifier elimination for a theory he calls A_L which is essentially the conjunction of our O1 to O4, S1, S3, and S4. Enderton (2001, Corollary 32B(b)) claims that $A_L = \mathrm{Th}(\mathbb{N}, <, \mathrm{s}, 0)$, but his theory cannot prove our axiom S2 (the discreteness of the order).

By using quantifier elimination, it is not hard to see that SA^- proves the induction schema.

Theorem 2.2. *For every formula* φ *in the language* $\mathcal{L}_{<,\mathrm{s}}$ *we have*

$$\mathsf{SA}^- \vdash \mathrm{Ind}_\varphi.$$

Proof. We shall prove that for every model M of SA^-, the only definable set which contains 0 and is closed under s is M itself. We say that $I \subseteq M$ is an *open interval* if there are $a, b \in M \cup \{\infty\}$ such that $I = \{x \in M\,;\, a < x < b\}$ and a set $X \subseteq M$ is called *basic* if it is a finite union of open intervals and singletons. As usual, an $\mathcal{L}$-theory T is called *o-minimal* or *order-minimal* if every $\mathcal{L}$-definable subset is basic.

We claim that SA^- is an o-minimal theory: Let $(M, 0, <, \mathrm{s}) \models \mathsf{SA}^-$ and $X \subseteq M$ be $\mathcal{L}_{<,\mathrm{s}}$-definable; by Lemma 2.1, SA^- has quantifier elimination and therefore, X is definable by a quantifier-free $\mathcal{L}_{<,\mathrm{s}}$-formula. We observe that sets definable by atomic formulae are either open intervals or points, hence basic; we furthermore observe that the basic sets are closed under finite intersections and complements. Thus all sets definable by quantifier-free formulae are basic.

By Theorem 1.5, in order to show induction, it is enough to show that the only non-empty $\mathcal{L}_{<,\mathrm{s}}$-definable cut of M is M itself. Suppose X is an $\mathcal{L}_{<,\mathrm{s}}$-definable cut in M. By o-minimality, we have that $X = I_0 \cup \ldots \cup I_n$ where for every $0 \leq i \leq n$, the set I_i is either an open interval (a_j, b_j) or a singleton $\{b_j\}$. Towards a contradiction, let $y \in M$ be such that $y \notin X$. We define $L := X \cap \mathrm{IS}(y)$ and $R := X \cap \mathrm{FS}(y)$, i.e., $X = L \cup R$. Note that there is $J \subseteq \{0, \ldots, n\}$ such that $L = \bigcup_{j \in J} I_j$ and that for $j \in J$, we have that $b_j \in M$. Let $m := \max\{b_j\,;\, j \in J\}$.

Case 1. $m \in L$. Then, since $L \subseteq X$, $m \in X$, but X is closed under successors, and so $\mathrm{s}(m) \in R$. But then $m < y < \mathrm{s}(m)$ which contradicts axiom S2.

Case 2. $m \notin L$. Then there is some $j \in J$ with $I_j = (a_j, m)$. By axiom S1, we find $m' \in I_j \subseteq X$ such that $\mathrm{s}(m') = m$. Once more, since X is closed under successors, $m \in R$, but this yields $m < y < \mathrm{s}(m)$ which contradicts axiom S2. $\square$

In particular, this means that SA and SA^- axiomatize the same theory:

Corollary 2.3. *Let M be a structure in the language $\mathcal{L}_{<,\mathrm{s}}$. Then $M \models \mathsf{SA}$ if and only if $M \models \mathsf{SA}^-$.*

Visser asked whether there is a reasonable finitely axiomatised theory that satisfies full induction (preferably in the full language of arithmetic); it is known that such a theory cannot be sequential (cf. Pudlák, 1985; Visser, 2010, for more on sequentiality). By Corollary 2.3, SA is a finitely axiomatised theory that satisfies full induction (and is not sequential).

Corollary 2.4. *A linear order L is the order type of a model of SA if and only if there is a linear order A such that $L \cong \mathbb{N} + \mathbb{Z} \cdot A$.*

Proof. By Corollary 2.3, it is enough to show that a model satisfies SA^- in order to get full SA. We already observed that the forward direction holds in § 1.4 (the linear order A is the quotient structure $M/{\sim}$ with the least element removed). For the other direction, if A is a linear order then $\mathbb{N} + \mathbb{Z} \cdot A$ can be easily made into an SA^- model by defining $\mathrm{s}(n) := n + 1$ and $\mathrm{s}(z, a) := (z + 1, a)$. □

3 Models based on generalised formal power series

Generalised formal power series, introduced by Levi-Civita, are a generalisation of polynomials over a ring: while polynomials only have natural number exponents, generalised formal power series allow exponents from any ordered additive abelian group. For an introduction to the theory of generalised formal power series, cf. (Fuchs, 1963). In this section, we shall adapt the classical theory of generalised formal power series to our context. In particular, we shall show how generalised power series can be used as a tool in building non-standard models of Pr^- and PA^-, and even Pr.

Definition 3.1. Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered group. A function $f : \Gamma \to B \cup \mathbb{Z}$ is a *non-negative formal power series* on B with exponents in Γ if $\operatorname{supp}(f)$ is reverse well-founded, for all $a \in \Gamma \setminus \{0\} f(a) \in B$, $f(0) \in \mathbb{Z}$, and $f(\textsc{lt}(f)) \geq 0$. We shall denote by $B(X^\Gamma)$ the set of *non-negative formal power series* with *base* B and *exponent* Γ.

We think of $f \in B(X^\Gamma)$ as the formal sum $\sum_{a \in \operatorname{supp}(f)} f(a) X^a$ and define order and additive structure on $B(X^\Gamma)$ according to this algebraic intuition:

Definition 3.2. Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered group. We define

$$(B(X^\Gamma), 0, <, \mathrm{s}, +)$$

to be the structure where $<$ is the anti-lexicographic order, i.e., $f < g$ if and only if $f \neq g$ and the biggest $a \in \Gamma$ such that $f(a) \neq g(a)$ is such that $f(a) < g(a)$, given $f, g \in \mathbb{Z}(X^\Gamma)$, we define $(f + g)(a) = f(a) + g(a)$, we interpret 0 as the constant 0 function and finally we define $\mathrm{s}(f)$ as $f + \mathbf{1}$ where $\mathbf{1}(0) = 1$ and $\mathbf{1}(a) = 0$ if $a \neq 0$.

Theorem 3.3. *Let $(\Gamma, 0, <)$ be a linear order with a minimum and $(B, 0, <, +)$ be an ordered abelian group. Then $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-.*

Proof. We want to show that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-. We shall first prove the closure of $B(X^\Gamma)$ under $+$. Let $f, g \in B(X^\Gamma)$. First of all note that by definition of $+$ we have $\mathrm{supp}(f+g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$ since $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-ordered so is $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ (any chain in $\mathrm{supp}(f) \cup \mathrm{supp}(g)$ contains a cofinal chain in $\mathrm{supp}(f)$ or $\mathrm{supp}(g)$). Therefore, $\mathrm{supp}(f+g)$ is reverse well-ordered. Moreover, $\textsc{lt}(f+g) = \max\{\textsc{lt}(f), \textsc{lt}(g)\}$. Indeed, if $\textsc{lt}(f) < \textsc{lt}(g)$ then trivially $\textsc{lt}(f+g) = \textsc{lt}(g)$, similarly for $\textsc{lt}(f) > \textsc{lt}(g)$ and $\textsc{lt}(f) = \textsc{lt}(g)$. Note that we have $f+g(\textsc{lt}(f+g)) \geq 0$. Again we have three cases $\textsc{lt}(f) < \textsc{lt}(g)$, $\textsc{lt}(f) > \textsc{lt}(g)$ and $\textsc{lt}(f) = \textsc{lt}(g)$. If $\textsc{lt}(f) < \textsc{lt}(g)$ then

$$\begin{aligned} f+g(\textsc{lt}(f+g)) = f+g(\textsc{lt}(g)) = \\ f(\textsc{lt}(g)) + g(\textsc{lt}(g)) = 0 + g(\textsc{lt}(g)) = g(\textsc{lt}(g)) \geq 0, \end{aligned}$$

similarly for $\textsc{lt}(f) > \textsc{lt}(g)$. If $\textsc{lt}(f) = \textsc{lt}(g)$ then

$$f+g(\textsc{lt}(f+g)) = f+g(\textsc{lt}(g)) = f(\textsc{lt}(f)) + g(\textsc{lt}(g)) \geq 0.$$

Finally, it is routine to check that all the axioms of Pr^- are satisfied by $(B(X^\Gamma), 0, <, \mathrm{s}, +)$. □

Let us consider a few instructive examples: If $\Gamma = \{0\} = 1$ and $B = \mathbb{Z}$ then $B(X^\Gamma) = \mathbb{Z}(X^1)$ and $(\mathbb{Z}(X^1), 0, <, \mathrm{s}, +)$ is isomorphic to the natural numbers. If $\Gamma = \{0,1\} = 2$ and $B = \mathbb{Z}$, then $B(X^\Gamma) = \mathbb{Z}(X^2)$ and $(\mathbb{Z}(X^2), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most 1 on $\mathbb{Z}$ with the standard order and operations. Similarly, if $\Gamma = \{0,1,2\} = 3$ and $B = \mathbb{Z}$, then $B(X^\Gamma) = \mathbb{Z}(X^3)$ and $(\mathbb{Z}(X^\Gamma), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most 2 over $\mathbb{Z}$ with the standard order and operations, and, more generally for every $0 < n \in \mathbb{N}$, if $\Gamma = n$ and $B = \mathbb{Z}$ then $(\mathbb{Z}(X^n), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials of degree at most $n-1$ over $\mathbb{Z}$ with the standard order and operations. Finally, by taking $\Gamma = \mathbb{N}$ and $B = \mathbb{Z}$ we have that $(\mathbb{Z}(X^\mathbb{N}), 0, <, \mathrm{s}, +)$ is isomorphic to the non-negative polynomials over $\mathbb{Z}$ with the standard order and operations. As mentioned in § 1.3, this means that the order type of $\mathbb{Z}(X^n)$ is O_n and the order type of $\mathbb{Z}(X^\mathbb{N})$ is O_ω.

Let $(\Gamma, 0, <, +)$ be an ordered commutative additive positive monoid and $(B, 0, 1, <, +, \cdot)$ be an ordered ring. We define a multiplicative structure over $B(X^\Gamma)$ as follows: for $f, g \in B(X^\Gamma)$ let $f \cdot g$ be the following function: if $a \in \Gamma$, then

$$(f \cdot g)(a) := \sum_{b+c=a} f(b) \cdot g(c).$$

We need to prove that this operation is well-defined:

Lemma 3.4. *Let* $(\Gamma, 0, <, +)$ *be an ordered commutative additive positive monoid and* $(B, 0, 1, <, +, \cdot)$ *be an ordered commutative ring. The multiplication over* $B(X^\Gamma)$ *is well-defined.*

Proof. It is enough to show that for each $a \in \Gamma$, there are only finitely many pairs $c, b \in \Gamma$ such that $c + b = a$ and $f(b) > 0$ and $g(c) > 0$. This follows from the fact that $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-founded: Assume that there is an infinite sequence $\langle c_n, b_n \rangle_{n\in\mathbb{N}}$ such that $c_n + b_n = a$, $f(b_n) \neq 0$, $g(c_n) \neq 0$, $c_n \neq c_{n+1}$ and $b_n \neq b_{n+1}$ for all $n \in \mathbb{N}$. We can build strictly increasing sequence either in $\mathrm{supp}(f)$ or in $\mathrm{supp}(g)$. Given a sequence $(s_n)_{n\in\mathbb{N}}$ we call an element s_n of the sequence a *spike* if for all $m > n$ we have $s_n > s_m$. Now consider the sequence $(c_n)_{n\in\mathbb{N}}$ either it has infinitely many spikes or there is n such that there are no spikes after n. If there are infinitely many spikes $(c_{n_m})_{m\in\mathbb{N}}$ in $(c_n)_{n\in\mathbb{N}}$ then they form an infinite strictly decreasing subsequence of $(c_n)_{n\in\mathbb{N}}$. Therefore, since $c_{n_m} + b_{n_m} = a$ and $c_{n_m} < c_{n_{m+1}}$, the sequence $(b_{n_m})_{m\in\mathbb{N}}$ is a strictly increasing sequence in $\mathrm{supp}(g)$. If there are only finitely many spikes there is trivially a strictly increasing subsequence in $(c_m)_{m\in\mathbb{N}}$. In both cases we obtain a contradiction since $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are reverse well-founded. □

The following theorem is the PA^--analogue of Theorem 3.3:

Theorem 3.5. *Let* $(\Gamma, 0, <, +)$ *be an ordered commutative additive positive monoid and* $(B, 0, 1, <, +, \cdot)$ *be an ordered commutative ring. Then the structure* $(B(X^\Gamma), 0, <, \mathrm{s}, +, \cdot)$ *is a model of* PA^-.

Proof. Since $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model Pr^-, we only need to prove that $B(X^\Gamma)$ is closed under $\cdot$ and that it satisfies the axioms M1 to M6. Let f and g be two functions in $B(X^\Gamma)$. We want to show $f \cdot g \in B(X^\Gamma)$. First of all note that since $\mathrm{supp}(f \cdot g) = \{a + b\,;\, a \in \mathrm{supp}(f) \text{ and } b \in \mathrm{supp}(g)\}$ then $\mathrm{supp}(f \cdot g)$ is reverse well-founded (by a similar argument as the one in the proof of Lemma 3.4). Note that since $\textsc{lt}(f \cdot g) = \textsc{lt}(f) + \textsc{lt}(g)$, we have that $(f \cdot g)(\textsc{lt}(f \cdot g)) = f(\textsc{lt}(b)) \cdot g(\textsc{lt}(c)) > 0$, since B is an ordered ring where products of positive elements are positive. Thus, $f \cdot g \in B(X^\Gamma)$.

It is again routine to check that the axioms M1 to M6 are satisfied by $B(X^\Gamma)$. □

We end this section by showing that if we require that B is divisible, then the resulting formal power series construction will give a non-standard model of Pr. This matches with Llewellyn-Jones's Theorem 4.1(ii) discussed in the next section.

Theorem 3.6. *Let* $(\Gamma, 0, <)$ *be a linearly ordered set with a minimum and* $(B, 0, <, +)$ *be a ordered divisible abelian group. Then* $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ *is a model of* Pr.

Proof. We already know that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^-. We shall show that $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ is a model of Pr^{D}, i.e., that for every natural number $n > 0$, the structure $(B(X^\Gamma), 0, <, \mathrm{s}, +)$ satisfies D_n.

Let $f \in B(X^\Gamma)$ and $0 < n \in \mathbb{N}$. First note that $\mathbb{Z}$ satisfies D_n for every $n > 0$ therefore there is $z \in \mathbb{Z}$ and a natural number $0 < m < n$ such that $f(0) = zn+m$. Moreover by divisibility of B for every $a \in \Gamma$ there is $b_a \in B$ such that $f(a) = b_a n$. Now, define $g \in B(X^\Gamma)$ as follows:

$$g(x) = \begin{cases} z & \text{if } x = 0, \\ b_x & \text{if } x > 0. \end{cases}$$

It is not hard to see that $f = \mathrm{s}^m(g \cdot n)$ as desired. $\square$

In particular note that if $B = \mathbb{Q}$ and $\Gamma = 2$, then $\mathbb{Q}(X^2)$ is a model of Pr of order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$. This model is well-known in the literature, cf., e.g., (Zoethout, 2015).

4 Presburger Arithmetic

Presburger arithmetic, the additive fragment of arithmetic, is closely related to ordered abelian groups. Llewellyn-Jones (2001) considers an integer version of Presburger arithmetic, allowing for additive inverses and gives an axiomatisation for this theory that we shall call $\mathsf{Pr}^{\mathbb{Z}}$. If $(M, 0, <, \mathrm{s}, +) \models \mathsf{Pr}^{\mathbb{Z}}$, then $(M, 0, <, +)$ is an ordered abelian group; Llewellyn-Jones calls these groups *Presburger groups*. Llewellyn-Jones proves in his integer setting that G is a Presburger group if and only if G is isomorphic to $\mathbb{Z} \cdot H$ where H is an ordered divisible abelian group (Llewellyn-Jones, 2001, §§ 3.1 & 3.2). In the following, we reformulate Llewellyn-Jones's approach in the standard setting of arithmetic (i.e., without additive inverses).

Theorem 4.1. *Let* M *be an* $\mathcal{L}_{<,\mathrm{s},+}$*-structure.*
(i) The structure M *is a model of* Pr^- *if and only if there is an ordered abelian group* G *such that* M *is isomorphic to the standard monoid over* G*, and*
(ii) the structure M *is a model of* Pr *if and only if there is an ordered divisible abelian group* G *such that* M *is isomorphic to the standard monoid over* G*.*

Proof. This proof is a reformulation of the characterisation of Presburger groups by Llewellyn-Jones (2001) to the standard setting.

For the forward direction of (i), it is enough to see that in $\mathbb{N}+\mathbb{Z}\cdot G^+$ all the axioms of Pr^- are trivially satisfied. For the other direction, if $M \models \mathsf{Pr}^-$ then by (the proof of) Corollary 2.4, the order type of M is $\mathbb{N}+\mathbb{Z}\cdot A$ for a linear order A consisting of the non-zero archimedean classes of M. For each $a \in A$, we define a formal *negative element* $-a$ such that the negative elements are all distinct from the elements of A and pairwise distinct. Then we define $-A := \{-a\,;\, a \in A\}$ and $G := -A \cup \{[0]\} \cup A$. For notational convenience, we define $-[0] := [0]$. We define an abelian group structure on G as follows:

1. For any $g \in G$, $g + [0] := [0] + g := g$.
2. If $a, b \in A$ are non-zero archimedean classes of M, then there is a unique $c \in A$ such that for all $x \in a$ and $y \in b$, we have that $x + y \in c$; define $a + b := b + a := c$ and $(-a) + (-b) := (-b) + (-a) := -c$.
3. If $a, b \in A$, $x \in A$, and $y \in b$ with $x < y$, then by $\divideontimes$, we find z such that $x + z = y$. Let c be the archimedean class of z, i.e., $c \in A \cup \{[0]\}$. Then $(-a) + b := b + (-a) := c$ and $a + (-b) := (-b) + a := -c$.

It is routine to check that $(G, 0, <, +)$ is an ordered abelian group and that M isomorphic to $\mathbb{N}+\mathbb{Z}\cdot G^+$. For (ii), all that is left to show that that divisibility of the group corresponds to the additional axioms D_n of Pr^{D}. □

Corollary 4.2 (Folklore)**.** *There is a model of* Pr *with order type* $\mathbb{N}+\mathbb{Z}\cdot\mathbb{R}$.

Proof. The real numbers $\mathbb{R}$ are an ordered divisible abelian group, so by Theorem 4.1 (i), there is a model of Pr with order type $\mathbb{N}+\mathbb{Z}\cdot\mathbb{R}^+$. The claim follows from the fact that $\mathbb{R}^+$ and $\mathbb{R}$ have the same order type. □

Corollary 4.3. *Let M be a non-standard model of* Pr. *Then M has order type $\mathbb{N}+\mathbb{Z}\cdot A$ where A is a dense linear order without endpoints.*

Proof. It is enough to observe that divisibility implies density and use Theorem 4.1. □

We can use Theorem 4.1 and the general theory of groupable linear orders to get a characterisation theorem for the order types of models of Pr^-. First let us recall a classical result about groupable linear orders; cf., e.g., (Rosenstein, 1982, Theorem 8.14):

Theorem 4.4. *A linear order $(L,<)$ is groupable if and only if there is an ordinal α and a densely ordered abelian group $(D,0,<,+)$ such that L has order type $\mathbb{Z}^\alpha \cdot D$.*

Corollary 4.5. *A structure M is a model of Pr^- if and only if there is an ordinal α and a densely ordered abelian group $(D,0,<,+)$ such that M has order type $\mathbb{N}+\mathbb{Z}\cdot(\mathbb{Z}^\alpha\cdot D)^+$.*

Proof. Follows from Theorems 4.1 & 4.4. □

As we have seen in § 3, the non-negative formal power series on $\mathbb{Z}$ with exponent 2 are isomorphic to the ordered abelian monoid of polynomials of degree at most 1 with integer coefficients. Moreover, by Theorem 3.3 (or Theorem 4.1), $(\mathbb{Z}(X^2),0,<,\mathrm{s},+)\models\mathsf{Pr}^-$. The next theorem shows that, in terms of order types, this is a lower bound for non-standard models of Pr^-.

Theorem 4.6. *Let M be a non-standard model of Pr^-. Then M has a submodel isomorphic to $(\mathbb{Z}(X^2),0,<,\mathrm{s},+)$.*

Proof. Let M be a non-standard model of Pr^- and $a\in M$ be a non-standard element of M. define the following mapping $\varphi:\mathbb{Z}(X^2)\to M$:

$$\varphi(f)=\begin{cases}\mathrm{s}^n(0) & \text{if } \textsc{lt}(f)=0 \text{ and } f(0)=n,\\ \mathrm{s}^m(na) & \text{if } \textsc{lt}(f)=1 \text{ and } f(1)=n,\ f(0)=m\geq 0,\\ b & \text{if } \textsc{lt}(f)=1 \text{ and } f(1)=n,\ f(0)=m<0 \text{ and } \mathrm{s}^{-m}(b)=na.\end{cases}$$

It is easy to see that φ is an order-preserving injection. □

Corollary 4.7. *Let M be a non-standard model of Pr^- then the order $\mathbb{N}+\mathbb{Z}\cdot\mathbb{N}$ can be embedded in the order type of M.*

Proof. As mentioned, $\mathbb{Z}(X^2)$ is the set of non-negative polynomials of degree at most 1 over $\mathbb{Z}$ and clearly has order type $\mathbb{N}+\mathbb{Z}\cdot\mathbb{N}$. The result then follows from Theorem 4.6. □

Corollary 4.8. *Every non-standard model of Pr^- has a proper non-standard submodel.*

Proof. By Theorem 4.6, it is enough to show that $\mathbb{Z}(X^2)$ has a non-standard submodel. Consider all polynomials with degree at most 1 and even leading terms, i.e.,

$$M:=\{2nX+z\in\mathbb{Z}(X^2)\,;\,n\in\mathbb{N},z\in\mathbb{Z}\}.$$

Clearly, this set is closed under s and $+$, so it is a substructure of $\mathbb{Z}(X^2)$. Since the only existential axiom of Pr^- is $\divideontimes$, it is enough to prove that M satisfies it. Let $f, g \in M$ such that $f < g$. Define $h(a) = g(a) - f(a)$. We want to show that $h \in M$. If $\textsc{lt}(f) = 0$ this is trivially true since $h(1) = g(1)$. If $\textsc{lt}(f) = 1$ then $f(1) = 2n$ and $g(1) = 2n'$ for some $n, n' \in \mathbb{N}$ such that $n < n'$. Then $h(1) = 2n' - 2n = 2(n' - n)$, therefore $h \in M$. The fact that $f + h = g$ follows trivially by the definition of $+$ in $\mathbb{Z}(X^2)$. $\square$

5 Peano Arithmetic

Theorem 4.1 tells us that every model $M \models \mathsf{PA}^-$ ($M \models \mathsf{PA}$) must have the order type $\mathbb{N} + \mathbb{Z} \cdot G^+$ where G is an ordered (divisible) abelian group. However, in the case of Peano Arithmetic, this cannot be a complete characterisation since Potthoff (1969) proved that no model of PA can have the order type $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$. The proof of Potthoff's theorem given by Bovykin and Kaye (2002, p. 5) easily generalises to PA^-:

Theorem 5.1. *Let M be a non-standard model of PA^- with order type $\mathbb{N}+\mathbb{Z}\cdot A$. If A is dense then there are $|M|$ many non empty disjoint intervals in A. In particular $A \neq \mathbb{R}$.*

Proof. Let $a \in M$ be non-standard. Consider the set $\{a_m \,;\, m \in M\}$ where $a_m = a \cdot m$ for every $m \in M$. By M6, this set has cardinality $|M|$. We shall now show that $\{(a_m, a_{\mathrm{s}(m)}) \,;\, m \in M\}$ forms a collection of non-empty disjoint intervals of size $|M|$:

By Lemma 1.2, $[a \cdot m] < [a \cdot \mathrm{s}(m)]$ for every $m \in M$. By density of A, the interval $([a_m], [a_{\mathrm{s}(m)}])$ is not empty in A. Now if $m < m'$, then by M6 we have $a \cdot \mathrm{s}(m) \le a \cdot m'$ and $[a \cdot \mathrm{s}(m)] \le [a \cdot m']$. Therefore $([a_m], [a_{\mathrm{s}(m)}]) \cap ([a_{m'}], [a_{\mathrm{s}(m')}]) = \varnothing$ as desired.

If $A = \mathbb{R}$, then the order type of M is $\mathbb{N} = \mathbb{Z} \cdot \mathbb{R}$ and hence $|M| = 2^{\aleph_0}$. Now the main claim of the theorem gives us an uncountable family of pairwise disjoint intervals in $\mathbb{R}$ which contradicts the countable chain condition of the real line. $\square$

Theorem 5.1 shows that the closure under multiplication adds more requirements on the order type of models of PA^-. One natural such requirement is the following:

Definition 5.2. Let L be a linear order. We say that L is *closed under finite products of initial segments* if for every $\ell \in L$ the order $\mathrm{IS}(\ell)^\omega$ embeds into $\mathrm{FS}(\ell)$.

Theorem 5.3. *Let M be a non-standard model of PA^- with order type $\mathbb{N}+\mathbb{Z}\cdot L$. Then L is closed under finite products of initial segments.*

Proof. As before, we assume that L is the set of non-zero archimedean classes of M. For every $\ell \in L$ choose a representative $r_\ell \in M$ such that $r_\ell \in \ell$ and $r_\ell > 0$. Let $\ell \in L$ be an element of the linear order L. We want to define an order embedding of $\mathrm{IS}(\ell)^\omega$ into $\mathrm{FS}(\ell)$. Fix some non-standard $a \in M$ such that $\ell \leq [a]$.

Clearly, $\mathrm{IS}(\ell)^\omega$ is order isomorphic to the functions from ω to $\mathrm{IS}(\ell)$ with finite support ordered anti-lexicographically. Consider the following function:

$$\varphi(f) = [\sum_{i \leq \mathrm{LT}(f)} r_{f(i)} \cdot a^{i+1}],$$

for every $f \in \mathrm{IS}(\ell)^\omega$. Note that since f has finite support, φ is well defined. Now we want to prove that φ is order-preserving. First we prove the following claim:

Claim 5.4. *For every $n > 0$ and every finite sequence $\langle \ell_0, \ldots, \ell_{n-1}\rangle$ of elements of $\mathrm{IS}(\ell)$ we have*

$$\sum_{i<n} r_{\ell_i} \cdot a^{i+1} < a^{n+1}.$$

Proof. By induction on n. For $n = 1$ we have $r_{\ell_0} \cdot a < a \cdot a$. For $n = n' + 1 > 1$ we have

$$\begin{aligned}
\sum_{i<n'+1} \cdot r_{\ell_i} \cdot a^{i+1} &= \sum_{i<n'} r_{\ell_i} \cdot a^{i+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\
&< a^{n'+1} + r_{\ell_{n'}} \cdot a^{n'+1} \\
&= a^{n'+1} \cdot (\mathrm{s}(0) + r_{\ell_{n'}}) < a^{n'+2}.
\end{aligned}$$

□

We want to prove that if $f < f'$ are two elements of $\mathrm{IS}(\ell)^\omega$ then $\varphi(f) < \varphi(f')$. Let $n \in \mathbb{N}$ be the biggest natural number such that $f(n) \neq f'(n)$. Since $f < f'$ we have $f(n) < f'(n)$, then $[r_{f(n)}] < [r_{f'(n)}]$.

Moreover since $n \leq \mathrm{LT}(f')$ we have

$$\sum_{n<i\leq \mathrm{LT}(f')} r_{f(i)} \cdot a^{i+1} = \sum_{n<i\leq \mathrm{LT}(f')} r_{f'(i)} \cdot a^{i+1}.$$

Therefore, by monotonicity of $+$ it is enough to prove that for every $n' \in \mathbb{N}$ we have

$$\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) < r_{f'(n)} \cdot a^{n+1}.$$

For $n = 0$ it is trivially true. For $n > 0$, we have

$$\begin{aligned}\sum_{i\leq n} r_{f(i)} \cdot a^{i+1} + \mathrm{s}^{n'}(0) &= \sum_{i<n} r_{f(i)} \cdot a^{i+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} + r_{f(n)} \cdot a^{n+1} + \mathrm{s}^{n'}(0) \\ &< a^{n+1} \cdot (r_{f(n)} + \mathrm{s}^{n'+1}(0)) \\ &< a^{n+1} \cdot r_{f'(n)},\end{aligned}$$

where we used Claim 5.4 in the first inequality. Therefore φ is order-preserving as desired. $\square$

Theorem 3.5 showed that the non-negative polynomials with integer coefficients $\mathbb{Z}(X^{\mathbb{N}})$ are a model of PA^-. In analogy to Theorem 4.6, we show that this provides a lower bound for the order type of non-standard models of PA^-:

Theorem 5.5. *Let M be a non-standard model of PA^-. Then there is a submodel of M isomorphic to $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$.*

Proof. Let M be a non-standard model of PA^- and $a \in M$ be a non-standard element of M. Let $f \in \mathbb{Z}(X^{\mathbb{N}})$; remember that if $\mathrm{supp}(f) \subseteq \{0, \dots, n\}$ and $\mathrm{LT}(f) = n$, then f can be thought of as a polynomial

$$f_n X^n + f_{n-1} X^{n-1} + \ldots + f_0$$

where $f_n > 0$ and $f_i \in \mathbb{Z}$ (for $0 \leq i < n$). We define the function

$$\varphi : \mathbb{Z}(X^{\mathbb{N}}) \to M : f \mapsto f_n a^n + f_{n-1} a^{n-1} + \ldots + f_0$$

where negative terms are unique interpreted by the fact that we have axiom $\divideontimes$. It is routine to check that φ is an embedding of $(\mathbb{Z}(X^{\mathbb{N}}), 0, <, \mathrm{s}, +, \cdot)$ into M. $\square$

Corollary 5.6. *Let M be a non-standard model of PA^-. Then the order type O_ω can be embedded in the order type of M. In particular $\mathbb{Z}(X^2)$ is not a model of PA^-.*

Proof. Since O_ω is the order type of the non-negative polynomials on $\mathbb{Z}$, this follows directly from Theorem 5.5. $\square$

Corollary 5.7. *Every non-standard model of PA^- has a proper non-standard submodel.*

Proof. As in the proof of Corollary 4.8, by Theorem 5.5, it is enough to check that $\mathbb{Z}(X^{\mathbb{N}})$ has a proper non-standard submodel. Consider the polynomials in which only terms with even exponent occur and observe that they are closed under addition and multiplication and that the structure satisfies $\circledast$. $\square$

We end this section by showing that our methods give an insight in the number of non-isomorphic order types of models of PA^- of a given cardinality. As we shall see, at least in the countable case, the situation is quite different from the one for the theories with induction, Pr and PA.

Lemma 5.8. *Let α and β be two positive ordinals. If $\mathbb{Z}(X^\alpha)$ is order isomorphic to $\mathbb{Z}(X^\beta)$ then $\alpha = \beta$.*

Proof. An easy induction shows that for every ordinal $\gamma > 0$, $\mathbb{Z}(X^\gamma)$ is order isomorphic to O_γ. Now we want to prove that if $0 < \alpha < \beta$ then O_β cannot be order embedded into O_α. First note that for every ordinal $0 < \alpha$ and for every order embedding φ of ω^α into $\mathbb{Z}^\alpha$ we have that φ is cofinal in $\mathbb{Z}^\alpha$. By induction on α. If $\alpha = 1$ or α is limit, the claim is trivially true. For $\alpha = \beta + 1$, let $\varphi : \omega^\beta \cdot \omega \to \mathbb{Z}^\alpha$ be an order embedding. Assume that there is $f \in \mathbb{Z}^\beta \cdot \mathbb{Z}$ such that for every $\gamma \in \omega^\beta \cdot \omega$ we have $\varphi(\gamma) < f$. Then $f = \langle g, z\rangle$ for some $g \in \mathbb{Z}^\beta$ and $z \in \mathbb{Z}$. Without loss of generality we can assume that z is the minimum such that f is an upper bound of φ. For every $\langle \gamma, n\rangle \in \omega^\beta \cdot \omega$ let us denote by $\langle g_{\langle \gamma,n\rangle}, z_{\langle \gamma,n\rangle}\rangle$ the image of $\langle \gamma, n\rangle$ under φ. Note that since for every $n \in \mathbb{N}$, the sequence $\{(\langle \gamma, n\rangle)\,;\, \gamma \in \omega^\beta\}$ is strictly increasing of order type ω^β, so it is its image. Moreover, since $z \in \mathbb{Z}$ and it is the minimum such that f is an upper bound of φ, there are $n \in \mathbb{N}$ and $\gamma \in \omega^\beta$ such that for every $\gamma' \in \omega^\beta$ if $\gamma < \gamma'$ we have $z_{\langle \gamma,n\rangle} = z_{\langle \gamma',n\rangle} = z$. Finally, since ω^β is additively indecomposable we have that $\{(g_{\langle \gamma',n\rangle})\,;\, \gamma < \gamma' \in \omega^\beta\}$ is a strictly increasing bounded sequence of order type ω^β in $\mathbb{Z}^\beta$. But this contradicts the inductive hypothesis.

Given what we have just proved, it is a routine induction to prove that for every $\alpha > 0$, α is the biggest ordinal such that ω^α can be embedded in O_α.

Therefore, for every $0 < \beta < \alpha$ we have that the order type of $\mathbb{Z}(X^\beta)$ is not isomorphic to the order type of $\mathbb{Z}(X^\alpha)$. $\square$

Theorem 5.9. *There are at least λ^+ non-isomorphic order types of models of PA^- of cardinality λ. Therefore, under GCH there are exactly 2^λ non isomorphic order types of models of PA^- of cardinality λ.*

Proof. Note that for every additively indecomposable ordinal α the structure $(\alpha, <, 0, \oplus)$ where $\oplus$ is the natural addition of ordinals, is an ordered commutative positive monoid. Since for every $\lambda < \alpha < \lambda^+$ we have $\omega^\alpha < \lambda^+$ then there are λ^+ many additively indecomposable ordinals smaller than λ^+. But then, since for every such ordinal ω^α we have that $(\mathbb{Z}(X)^{\omega^\alpha}, 0, <, \mathrm{s}, +, \cdot)$ is a model of PA^- of cardinality λ. Hence there are at least λ^+ non-isomorphic order types of models of PA^- of cardinality λ as desired. □

In particular, note that for $\lambda = \omega$, Theorem 5.9 gives us uncountably many non-isomorphic countable models of PA^- in stark contrast with the two order types of countable models of PA (by Cantor's theorem, $\mathbb{N}$ and $\mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$ are the only possible order types). Moreover, note that none of the order types generated by the proof of Theorem 5.9 satisfy the requirements of Corollary 4.3, and so they cannot be order types of models of Pr (nor of PA). Therefore, we have:

Corollary 5.10. *There are at least λ^+ non-isomorphic order types of models of PA^- of cardinality λ which are not order types of models of Pr or PA.*

6 Summary

As mentioned, one of the major open questions in this area is a complete characterisation of the order types of models of PA. For the theories SA and Pr^-, we were able to give complete characterisations in Corollaries 2.4 and 4.5; for the theories Pr and PA^-, we were able to give necessary conditions in Corollary 4.3 and Theorems 5.1 and 5.3, respectively. In particular, the negative results from §§ 3 & 4 imply:

Corollary 6.1. *There is no model of Pr (and hence, no model of PA) with order type O_2 or O_ω.*

Proof. We have that $O_2 = \mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ and $O_\omega = \mathbb{N} + \mathbb{Z} \cdot O_\omega$. Clearly, $\mathbb{N}$ and O_ω are not the positive parts of a densely ordered abelian group, so by Corollary 4.3, no model of Pr can have these order types. □

We are now in the position to combine our results to show the separation of the five theories mentioned in § 1.1 in terms of order types. In the following diagram, an arrow from a theory T to a theory S means "every order type that occurs in a model of T occurs in a model of S". The diagram is complete in the sense that the absence of an arrow means that no arrow can be drawn, i.e., "there is an order type of a model of T that cannot be an order type of a model of S".

$$\begin{tikzcd}
& \mathsf{SA} & \\
\mathsf{Pr}^- \arrow[ur] & & \mathsf{Pr} \arrow[ul] \arrow[ll] \\
\mathsf{PA}^- \arrow[u] & & \mathsf{PA} \arrow[u] \arrow[ll]
\end{tikzcd}$$

$\mathsf{SA} \not\twoheadrightarrow \mathsf{Pr}^-$ Follows from Corollary 2.4 and Corollary 4.7: $\mathbb{N} + \mathbb{Z}$ is an order type witnessing the separation.

$\mathsf{Pr}^- \not\twoheadrightarrow \mathsf{Pr}$ Follows from Theorem 3.3 and Corollary 6.1: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

$\mathsf{Pr}^- \not\twoheadrightarrow \mathsf{PA}^-$ Follows from Theorem 3.3 and Corollary 5.6: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{N}$ is an order type witnessing the separation.

$\mathsf{PA}^- \not\twoheadrightarrow \mathsf{Pr}$ Follows from Theorem 3.5 and Corollary 6.1: $\mathbb{N} + \mathbb{Z} \cdot O_\omega$ is an order type witnessing the separation.

$\mathsf{Pr} \not\twoheadrightarrow \mathsf{PA}^-$ Follows from Theorem 5.1 and Corollary 4.2: $\mathbb{N} + \mathbb{Z} \cdot \mathbb{R}$ is an order type witnessing the separation.

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