Edmonds' Branching Theorem in Digraphs without Forward-infinite Paths

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Abstract

Let D be a finite digraph, and let $V_0, \dots, V_{k-1}$ be nonempty subsets of $V(D)$. The (strong form of) Edmonds' branching theorem states that there are pairwise edge-disjoint spanning branchings $\mathcal{B}_0, \dots, \mathcal{B}_{k-1}$ in D such that the root set of $\mathcal{B}_i$ is V_i ($i = 0, \dots, k-1$) if and only if for all $\varnothing \neq X \subseteq V(D)$ the number of ingoing edges of X is greater than or equal to the number of sets V_i disjoint from X. As was shown by R. Aharoni and C. Thomassen in [1], this theorem does not remain true for infinite digraphs. Thomassen also proved that for the class of digraphs without backward-infinite paths, the above theorem of Edmonds remains true. Our main result is that for digraphs without forward-infinite paths, Edmonds' branching theorem remains true as well.

1 Notions and notation

The digraphs $D = (V, A)$ considered here may have multiple edges and arbitrary size. Loops are also allowed but are irrelevant to our subject. If $B \subseteq V$, then we write $D[B]$ for the subgraph of D spanned by B. For $X \subseteq V$ let $\mathsf{in}_D(X)$ and $\mathsf{out}_D(X)$ be the set of ingoing and outgoing edges respectively of X in D, and let $\varrho_D(X)$, $\delta_D(X)$ be their respective cardinalities. By a path, we mean a directed, possibly infinite, simple path (the repetition of vertices is not allowed). We denote by $\mathsf{start}(P)$ and $\mathsf{end}(P)$ the first and last vertex of the path P, if they exist. For an edge e from x to y, let $\mathsf{start}(e) = x$ and $\mathsf{end}(e) = y$. For $X, Y \subseteq V$, let $\mathsf{e}_D(X, Y) = \{e \in A : \mathsf{start}(e) \in X,\ \mathsf{end}(e) \in Y\}$; for singletons we write $\mathsf{e}(x, y)$ instead of $\mathsf{e}(\{x\}, \{y\})$. We say that the path P *goes from* X *to* Y if $V(P) \cap X = \{\mathsf{start}(P)\}$ and $V(P) \cap Y = \{\mathsf{end}(P)\}$ ($\mathsf{start}(P) = \mathsf{end}(P)$ is allowed). We call $\min\{\varrho_D(X) : \varnothing \neq X \subseteq V \setminus \{r\}\}$ the edge-connectivity of D from r, and D is κ-edge-connected from r if this cardinal is at least κ.

A digraph is an *arborescence* with root vertex r if it is a directed tree such that all vertices are reachable from r. A digraph is a *branching* with root set W if its weakly connected components are arborescences and the vertex set W consists of the roots of these arborescences. $\mathcal{B}$ is a *k-branching*

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in D iff it is a k-tuple $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{k-1})$ such that the $\mathcal{B}_i = (V_i, A_i)$'s are edge-disjoint branchings in D (not necessarily with the same root sets), and we let $D \setminus\!\!\setminus \mathcal{B} = (V, A \setminus \cup_{i<k} A_i)$. If F is a branching and P is a path such that $V(F) \cap V(P) = \{\mathsf{start}(P)\}$, then we denote by $F + P$ the branching $(V(F) \cup V(P), A(F) \cup A(P))$.

2 Introduction

Edmonds proved in [3] his famous theorem (now called the *weak form of Edmods' branching theorem*) which states that if a finite digraph is k-edge-connected from a vertex r for some $k \in \mathbb{N}$, then it has k edge-disjoint spanning arborescences rooted at r. He also proved a generalization of this (called the *strong form of Edmods' branching theorem*; see [4] p. 349 Theorem 10.2.1) which states the following. If D is a finite digraph and $V_0, \dots, V_{k-1} \subseteq V(D)$, then there are pairwise edge-disjoint spanning branchings $\mathcal{B}_0, \dots, \mathcal{B}_{k-1}$ in D such that the root set of $\mathcal{B}_i$ is V_i $(i = 0, \dots, k-1)$ if and only if all $\varnothing \neq X \subseteq V(D)$ has at least $|\{i < k : V_i \cap X = \varnothing\}|$ ingoing edges. L. Lovász gave a new elegant proof for Edmods' branching theorem in [9], and his techniques opened the door for further generalizations such as [7], [5], [2] and [8]. Infinite generalizations have been obstructed by a negative result of R. Aharoni and C. Thomassen [1]. They constructed, for any $k \in \mathbb{N}$, a countably-infinite, locally finite, simple graph G such that G has a k-connected orientation but has vertices u, v such that deleting the edges of an arbitrary path between u and v makes the remaining graph disconnected.

Thomassen showed (unpublished) that if $D = (V, A)$ does not contain backward-infinite paths and is k-edge-connected from r for some $k \in \mathbb{N}$, then it has k edge-disjoint spanning arborescences rooted at r. The main idea of his proof is the following: construct first a spanning subgraph $D' = (V, A')$ of D such that D' is also k-edge-connected from r and all vertices of D' have finite indegrees. After that, one can build the desired arborescences in D' using the finite version of the theorem and compactness arguments. Thomassen's proof also works for the strong form of the Edmonds' branching theorem. Our main result is that disallowance of forward-infinite paths instead of backward-infinite paths is also sufficient. Our proof uses techniques very different from Thomassen's proof.

There is a general approach in finite combinatorics based on separating by "tight" sets to smaller subproblems and handling of these by induction independently. This approach works for example for Menger's theorem and for Edmonds' branching theorem but obviously can not be used directly to infinite generalizations because it is possible that the subproblems have the same size as the original. Even so we will define the notion of "tightness" in the context of Edmonds' branching theorem and it will play key role in our proof. An other proof for the finite case given by Lovász in [9] makes it possible (even without the restriction about infinite paths) to create edge-disjoint branchings with the prescribed root sets where all of them have infinitely many vertices. Unfortunately using Lovász's approach we can not guarantee that the resulting branchings will be spanning branching (not even in the countable case) because we can not control that which vertex do we extend a branching with. This controllability will be essential in our proof to ensure conditions after limit steps in our recursive construction.

3 Main result

In this section, we state and prove our main result. Instead of packing branchings with prescribed root sets, we formulate this result in a formally more general (but in fact equivalent) form, in which we want to extend some initial edge-disjoint branchings to edge-disjoint spanning branchings without changing their root sets. If these initial branchings have no edges, then we get back the "prescribed root sets"-approach.

Theorem 1. *Let $D = (V, A)$ be a digraph, $k \in \mathbb{N}$ and $\mathcal{B}_i = (V_i, A_i)$ $(i < k)$ edge-disjoint branchings in D and let $D \mathbin{\backslash\!\backslash} \mathcal{B} = (V, A \setminus \cup_{i<k} A_i)$. Suppose that $D \mathbin{\backslash\!\backslash} \mathcal{B}$ does not contain forward-infinite paths. Then the branchings can be extended to edge-disjoint spanning branchings of D without changing their root sets if and only if*

$$\forall X \ (\varnothing \neq X \subseteq V \Longrightarrow \varrho_{D \mathbin{\backslash\!\backslash} \mathcal{B}}(X) \geq |\{i < k : V_i \cap X = \varnothing\}|). \tag{1}$$

If there is an $r \in V$ such that $V_i = \{r\}$ for all $i < k$, then we get the following special case.

Corollary 2. *Let the digraph D be k-edge-connected from the vertex r for some $k \in \mathbb{N}$, and suppose that there are no forward-infinite paths in D. Then there are k edge-disjoint spanning arborescences in D rooted at r.*

Remark 3. Our proof of Theorem 1 also works in a more general case when there is no restriction on the quantity of the initial branchings, but all vertices belong to all but finitely many of these branchings.

Proof of Theorem 1. The necessity of condition (1) is obvious, so we show only that it is sufficient. To do so, we need the following lemma.

Lemma 4. *For any $j < k$ and $v \in V \setminus V_j$, there is a path P in $D \mathbin{\backslash\!\backslash} \mathcal{B}$ from V_j to v such that condition (1) holds for D and the k-branching $\mathcal{B}'$, where $\mathcal{B}'_i = \begin{cases} \mathcal{B}_i + P & \text{if } i = j \\ \mathcal{B}_i & \text{otherwise} \end{cases}$.*

Without loss of generality, it is enough to prove Lemma 4 for $j = 0$, because the role of the initial branchings are symmetric. Before the proof, we need to devolp some basic tools in the spirit of Lovász's proof for the finite version of the theorem in [9].

3.1 Basic tools

We will prove here some facts which are known from finite branching-packing techniques and remain true with the same proof in the infinite case. In this subsection, we fix a digraph $D = (V, A)$ and a k-branching $\mathcal{B}$ $(\mathcal{B}_i = (V_i, A_i),\ i = 0, \dots, k-1)$ of D, that satisfy condition (1).

Call a set $\varnothing \neq X \subseteq V$ **tight** (with respect to $\mathcal{B}$), if $\varrho_{D \mathbin{\backslash\!\backslash} \mathcal{B}}(X) = |\{i < k : V_i \cap X = \varnothing\}|$; and **dangerous**, if it is tight and $X \cap V_0 \neq \varnothing$. For example, V itself is dangerous. It is easy to see that if $e \in \mathsf{out}_{D \mathbin{\backslash\!\backslash} \mathcal{B}}(V_0)$, then the extension $\mathcal{B}'_0 = \mathcal{B}_0 + e$ violates condition (1) if and only if e is an ingoing edge of some dangerous set.

Proposition 5. *If X, Y are dangerous and $X \cap Y \neq \varnothing$, then $X \cap Y$ is also dangerous.*

Proof: Let $s : \mathcal{P}(V) \to \mathbb{N}$, $s(X) = |\{i < k : V_i \cap X = \varnothing\}|$. Then s is supermodular i.e., for $X, Y \subseteq V$, we have $s(X) + s(Y) \leq s(X \cup Y) + s(X \cap Y)$. Indeed, let $i < k$ be arbitrary. If $V_i \cap X = \varnothing$ and $V_i \cap Y = \varnothing$, then $V_i \cap (X \cup Y) = \varnothing$ and $V_i \cap X \cap Y = \varnothing$, so V_i's contribution to both sides of the inequality is 2. If $V_i \cap X = \varnothing$ and $V_i \cap Y \neq \varnothing$, then V_i's contribution to both sides is 1. Observe that equality holds if and only if there is no V_i such that $V_i \cap X \neq \varnothing$, $V_i \cap Y \neq \varnothing$ but $V_i \cap X \cap Y = \varnothing$. Let $p(X) = \varrho_{D \setminus\setminus \mathcal{B}}(X) - s(X)$ (for an infinite cardinal κ and $n \in \mathbb{N}$ let $\kappa - n = \kappa$). Then condition (1) is equivalent with the requirement $p(X) \geq 0$ for all $X \neq \varnothing$, and the tightness of X means $p(X) = 0$. The function $\varrho_{D \setminus\setminus \mathcal{B}}$ is submodular, therefore so is p i.e. $p(X) + p(Y) \geq p(X \cup Y) + p(X \cap Y)$ holds for all $X, Y \subseteq V$. Let X, Y be dangerous, and $X \cap Y \neq \varnothing$. Then by submodularity and by condition (1), we get

$$0 + 0 = p(X) + p(Y) \geq p(X \cup Y) + p(X \cap Y) \geq 0 + 0,$$

so $X \cup Y$ and $X \cap Y$ are tight. Therefore $s(X) + s(Y) = s(X \cup Y) + s(X \cap Y)$. By the observation about the function s, we may conclude from $X \cap V_0 \neq \varnothing$, $Y \cap V_0 \neq \varnothing$ that $X \cap Y \cap V_0 \neq \varnothing$, so $X \cap Y$ is dangerous. ●

Proposition 6. *Let B be a dangerous set. Then for any $w \in B$, there is a path R from $V_0 \cap B$ to w in $(D \setminus\setminus \mathcal{B})[B]$.*

Proof: Let B' be the set of vertices which are reachable from $V_0 \cap B$ in $(D \setminus\setminus \mathcal{B})[B]$. Suppose, that $B' \neq B$. Then $B \setminus B'$ violates condition (1), which is a contradiction. ●

Proposition 7. *For all $w \in V$, there is a system of edge-disjoint paths $\{P_i\}_{i<k}$ in $D \setminus\setminus \mathcal{B}$ such that P_i goes from V_i to w.*

Proof: We extend $D \setminus\setminus \mathcal{B}$ to H by adding new vertices and edges (see figure 1). Let $V(H) = V \cup \{s\} \cup \{v_i\}_{i<k}$, $|\mathsf{e}_H(s, v_i)| = 1$ $(i < k)$ and $|\mathsf{e}_H(v_i, u)| = \aleph_0$ $(i < k, u \in V_i)$. If there are k edge-disjoint paths from s to w in H, then we are done. Suppose, seeking a contradiction, that there are not. By Menger's theorem, there is a $w \in X \subseteq V(H) \setminus \{s\}$ with $\varrho_H(X) < k$. Let $l = \left|\{v_i\}_{i<k} \setminus X\right|$. Note that $0 < l$, otherwise $sv_i \in \mathsf{in}_H(X)$ $(i < k)$ and hence $k \leq \varrho_H(X)$ would follow. Since there are infinitely many parallel edges, $X \cap V$ is disjoint from at least l branchings. Otherwise $\varrho_H(X) = \varrho_{D \setminus\setminus \mathcal{B}}(X \cap V) + (k - l)$, so $\varrho_{D \setminus\setminus \mathcal{B}}(X \cap V) = l + (\varrho_H(X) - k) < l$, but then $X \cap V$ violates condition (1) in D giving us a contradiction. ●

Corollary 8. *Let $B_1 \subseteq B_0$ be dangerous sets and let $\varrho_{D \setminus\setminus \mathcal{B}}(B_0) = \varrho_{D \setminus\setminus \mathcal{B}}(B_0) = l \geq 1$. Let $s_j = \mathsf{end}(e_j)$, where $\{e_1, \ldots, e_l\} = \mathsf{in}_{D \setminus\setminus \mathcal{B}}(B_0)$. Then there is a system of edge-disjoint paths $\{P_j\}_{j=1}^{l}$ in $(D \setminus\setminus \mathcal{B})[B_0]$ such that P_j goes from s_j to B_1. Such a path system necessarily contains all of the elements of $\mathsf{e}_{D \setminus\setminus \mathcal{B}}(B_0, B_1 \setminus B_0)$, and the multiset of the endpoints of their elements is $\{\mathsf{end}(e) : e \in \mathsf{in}_{D \setminus\setminus \mathcal{B}}(B_1)\}$.*

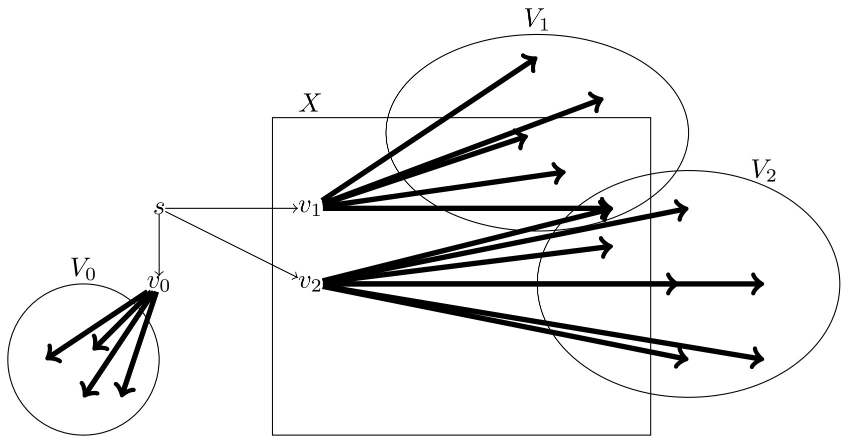

Figure 1: The construction of H and the cut X from the proof above in the case $k = 3$, $l = 1$. (Thick arrows stand for countably infinite parallel edges).

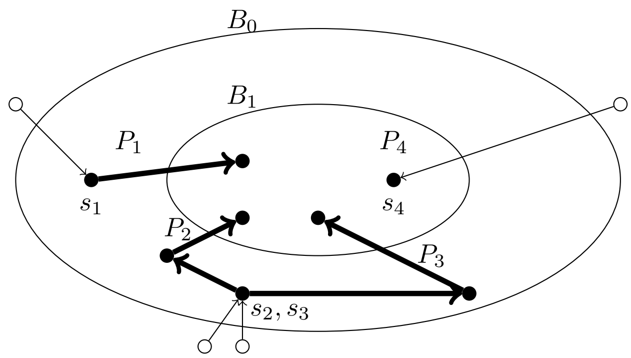

Figure 2: Corollary 8 in the case $l = 4$. We thickened the desired path system $\{P_j\}_{j=1}^{4}$. In this example, $s_2 = s_3$ and P_4 consists of the vertex s_4.

Proof: B_0 is disjoint from exactly l many of the sets V_i because B_0 is dangerous and $\varrho_{D \setminus\!\!\setminus \mathcal{B}}(B_0) = l$. Without loss of generality, we may assume that these sets are $V_1, V_2, \ldots, V_l$. By Proposition 7, there is a system of edge-disjoint paths $\{P'_j\}_{j=1}^{l}$ in $D \setminus\!\!\setminus \mathcal{B}$ such that P'_j goes from V_j to B_1. Note that such a path system necessarily contains all the edges in $\mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}}(B_0) \cup \mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}}(B_1)$, and all the paths enter B_0 exactly once. By deleting the initial segments of the paths P'_j that are not in B_0, we get the desired path system. ●

3.2 Proof of the main Lemma

Now we are able to to prove Lemma 4.

Proof: Assume, seeking a contradiction, that Lemma 4 is false and $v \in V \setminus V_0$ witnesses this. We will construct three sequences: $\mathcal{B}_0^n = (V_0^n, A_0^n), B_n, e_n$ $(n \in \mathbb{N})$. Let $B_0 = V$, $\mathcal{B}_0^0 = \mathcal{B}_0$ and let e_0 be an arbitrary edge. We will denote the k-branching $(\mathcal{B}_0^n, \mathcal{B}_1, \ldots, \mathcal{B}_{k-1})$ by $\mathcal{B}^n$.

Let Q be a path from V_0 to v in $D \setminus\!\!\setminus \mathcal{B}$ (such a path exists by Proposition 6). Let u be the last vertex of Q for which there is a path R from V_0 to u in $D \setminus\!\!\setminus \mathcal{B}$ such that $\mathcal{B}_0^1 \stackrel{\text{def}}{=} \mathcal{B}_0 + R$ does not violate condition (1). Since u cannot be the last vertex of Q, there is a unique outgoing edge e_1 of u which is in Q (see figure 3). The extension $\mathcal{B}_0^1 + e_1$ violates condition (1) because of the choice of u, and thus $e_1 \in \mathsf{in}_{D\setminus\!\!\setminus\mathcal{B}^1}(B_1)$ where B_1 is a set which is dangerous with respect to $\mathcal{B}^1$.

Our plan is to continue by doing the same but inside B_1. Let Q_1 be an arbitrary path from $V_0^1 \cap B_1$ to $\mathsf{end}(e_1)$ in $(D \setminus\!\!\setminus \mathcal{B}^1)[B_1]$ (such a path exists by Proposition 6). Let u_1 be the last vertex of Q_1 for which there is a path R_1 in $(D \setminus\!\!\setminus \mathcal{B}^1)[B_1]$ from $V_0^1 \cap B_1$ to u_1 such that $\mathcal{B}_0^2 \stackrel{\text{def}}{=} \mathcal{B}_0^1 + R_1$ does not violate condition (1). Since $u_1 \neq \mathsf{end}(e_1)$, there is a unique outgoing edge e_2 of u_1 which is in Q_1. The extension $\mathcal{B}_0^2 + e_2$ violates condition (1) because of the choice of u_1, thus $e_2 \in \mathsf{in}_{D\setminus\!\!\setminus\mathcal{B}^2}(B_2)$, where $B_2 \subsetneq B_1$ is a set which is dangerous with respect to $\mathcal{B}^2$ (if $B_2 \not\subseteq B_1$, then by Proposition 5, we may replace B_2 with $B_2 \cap B_1$).

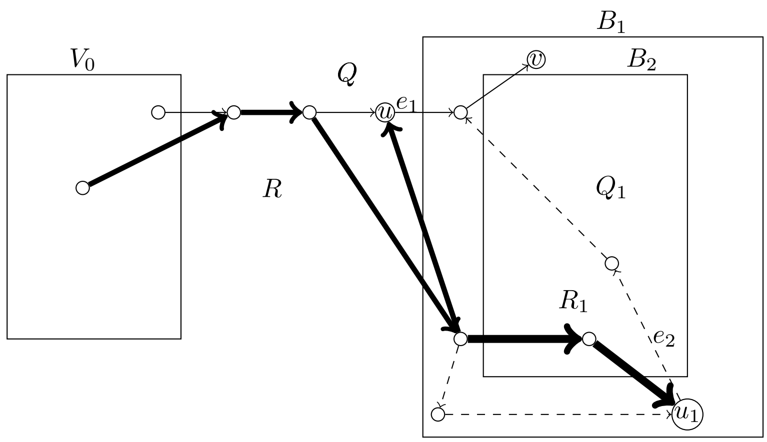

Figure 3: The process described above. The path Q is represented with a normal line, R with a thick line, Q_1 with a dashed line and R_1 with a very thick line.

By continuing the process recursively we get the desired sequences with the following properties: for all $n \in \mathbb{N}$:

1. $B_{n+1} \subsetneq B_n$,
2. (a) $\mathcal{B}^n$ satisfies condition (1),
   (b) the sets $B_0, \dots, B_n$ are dangerous with respect to $\mathcal{B}^n$,
   (c) $\mathsf{in}_{D\setminus\!\!\setminus\mathcal{B}^n}(B_n) = \mathsf{in}_{D\setminus\!\!\setminus\mathcal{B}^{n+1}}(B_n)$,
3. $e_{n+1} \in \mathsf{e}_{D\setminus\!\!\setminus\mathcal{B}^{n+1}}(B_n \setminus B_{n+1}, B_{n+1})$, (and so the edges e_{n+1} ($n \in \mathbb{N}$) are pairwise distinct).

By throwing away the first finitely many elements of the sequences constructed above and reindexing them, we may assume that all the members of the monotone decreasing sequence B_n are disjoint from exactly the same, say l many, of sets among $V_1, \dots, V_{k-1}$. Without loss of generality we may assume that these sets are $V_1, V_2, \dots, V_l$. Note that $l \geq 1$ because B_n is dangerous with respect to $\mathcal{B}^n$ and $\varrho_{D\setminus\!\!\setminus\mathcal{B}^n}(B_n) \geq 1$ because $e_n \in \mathsf{in}_{D\setminus\!\!\setminus\mathcal{B}^n}(B_n)$.

For $n \in \mathbb{N}$, let $\{P_j^n\}_{j=1}^l$ be a system of obtained by applying Corollary 8 with $B_{n+1} \subsetneq B_n$ and $\mathcal{B}^{n+1}$. Note that $e_{n+1} \in \mathsf{e}_{D \setminus\!\setminus \mathcal{B}^{n+1}}(B_n \setminus B_{n+1}, B_{n+1}) \subseteq \bigcup_{j=1}^l A(P_j^n)$. The multisets $\{\mathsf{end}(P_j^n)\}_{j=1}^l$ and $\{\mathsf{start}(P_j^{n+1})\}_{j=1}^l$ are equal (they are $\{\mathsf{end}(e) : e \in \mathsf{in}_{D \setminus\!\setminus \mathcal{B}^n}(B_n) = \mathsf{in}_{D \setminus\!\setminus \mathcal{B}^{n+1}}(B_n)\}$), so we can concatenate the path systems $\{P_j^n\}_{j=1}^l$ and $\{P_j^{n+1}\}_{j=1}^l$ for all n. Thus, we obtain a system of edge-disjoint paths $\{P_j\}_{j=1}^l$ in $D \setminus\!\setminus \mathcal{B}$ (see figure 4) such that $\{e_n\}_{n=1}^\infty \subseteq \bigcup_{j=1}^l A(P_j)$, and therefore at least one of them is forward-infinite, which contradicts the conditions of Theorem 1. ■

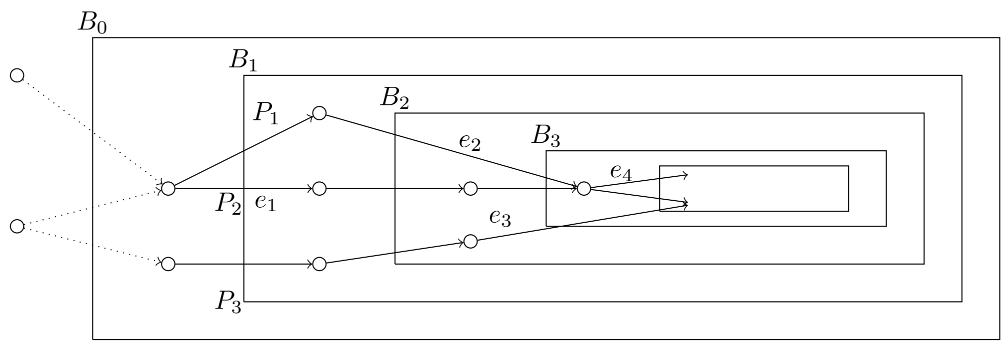

Figure 4: The (initial segment of) path system $\{P_j\}_{j=1}^l$ in the case $l = 3$.

3.3 Proof of the Theorem

Now, we continue the proof of Theorem 1. If $v \in V$, then by Lemma 4, we can extend the branchings, without violating condition (1), with finitely many new vertices and edges such that all of these extensions contain v. In the countable case, we can construct the desired spanning branchings by the following recursion. In the n-th step, do the extensions above with the branchings after the previous step and with the next vertex v_n where $V = \{v_n\}_{n=0}^\infty$. In the uncountable case, we have to be more careful because we can not avoid limit steps, and we need to assure that we do not violate condition (1) in these steps as well. The easy trick to handle this is that if we extend in one step one of the branchings with some vertex v, then before the next limit step we put v into all the branchings which missed it.

Let us make this precise. Let $V \overset{\text{def}}{=} \{v_\alpha\}_{\alpha<\lambda}$, where $\lambda = |V|$. We extend the branchings by transfinite recursion on λ. Denote by $\mathcal{B}_i^\alpha = (V_i^\alpha, A_i^\alpha)$ the branching which we get from $\mathcal{B}_i$ after the α-th step, and let $\mathcal{B}^\alpha = (\mathcal{B}_0^\alpha, \mathcal{B}_1^\alpha, \dots, \mathcal{B}_{k-1}^\alpha)$ for $\alpha \leq \lambda$.

Let $\mathcal{B}_i^0 = \mathcal{B}_i$ $(i < k)$.
If $\alpha < \lambda$ is a limit ordinal, then let $\mathcal{B}_i^\alpha = (\bigcup_{\beta<\alpha} V_i^\beta, \bigcup_{\beta<\alpha} A_i^\beta)$.
If $\alpha = \beta + 1$ where $\beta < \lambda$ is a limit ordinal and $\mathcal{B}^\beta$ satisfies condition (1), then add v_β to all of the branchings $\{\mathcal{B}_i^\beta\}_{i<k}$ by using Lemma 4 repeatedly. Denote the resulting k-branching by $\mathcal{B}^\alpha$.
If $\alpha = \beta + 2$ where $\beta < \lambda$ is an arbitrary ordinal and $\mathcal{B}^{\beta+1}$ satisfies condition (1) and the set $N^\beta \overset{\text{def}}{=} \{v_{\beta+1}\} \cup \bigcup_{i<k} V_i^{\beta+1} \setminus V_i^\beta$ is finite, then add the elements of N^β one by one to all of the branchings $\{\mathcal{B}_i^{\beta+1}\}_{i<k}$ by using Lemma 4 repeatedly. Denote the resulting k-branching $\mathcal{B}^\alpha$.

Proposition 9. *The transfinite recursion above does not stop before the λ-th step.*

Proof: Suppose, seeking a contradiction, that it does. The limit steps are well defined. At successor steps, we do not violate condition (1), and we have extended the branchings with only finitely many new vertices and edges. Thus if $\mathcal{B}^{\gamma+1}$ is well defined for some $\gamma < \lambda$, then so is $\mathcal{B}^{\gamma+2}$. Hence the first step where the recursion can not be continued is necessarily a successor of a limit ordinal β. But then β is the first ordinal such that $\mathcal{B}^{\beta}$ violates condition (1). Consider the function $s_\beta(X) = |\{i < k : V_i^\beta \cap X = \varnothing\}|$, and fix an arbitrary $\varnothing \neq X \subseteq V$. If $\mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}^\beta}(X) = \mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}}(X)$, then $\varrho_{D \setminus\!\!\setminus \mathcal{B}^\beta}(X) = \varrho_{D \setminus\!\!\setminus \mathcal{B}}(X) \geq s_0(Y) \geq s_\beta(X)$. Otherwise there exists an $e \in \mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}}(X) \setminus \mathsf{in}_{D \setminus\!\!\setminus \mathcal{B}^\beta}(X)$, and so there is an $i_0 < k$ such that $e \in A_{i_0}^\beta$. Let $\gamma < \beta$ be the smallest ordinal such that $e \in A_{i_0}^\gamma$. Then γ is a successor ordinal, so by the recursion we have $\mathsf{end}(e) \in V_i^{\gamma+1}$ $(i < k)$. Thus $\mathsf{end}(e) \in V_i^\beta$ $(i < k)$, and therefore $\varrho_{D \setminus\!\!\setminus \mathcal{B}^\beta}(Y) \geq 0 = s_\beta(X)$. Hence no set X violates condition (1) with respect to $\mathcal{B}^\beta$, which is a contradiction. ●

Finally, $\mathcal{B}_i^\lambda$ $(i = 0, \dots, k-1)$ are the desired spanning branchings. □

4 A conjecture about packing infinitely many branchings

We show, by a counterexample, that the finiteness of the number of initial branchings is necessary in Theorem 1, and formulate a conjecture with a very natural condition about paths (which is a strengthening of the condition (1)) motivated by this counterexample.

We construct a digraph D and a system of edge-disjoint branchings $\mathcal{B} = \langle \mathcal{B}_n : n \in \mathbb{N} \rangle$ of D such that $D \setminus\!\!\setminus \mathcal{B}$ does not contain infinite paths and the system satisfies condition (1), but the desired extensions of the branchings do not exist. Let $D = (V, A)$ where $V = \{r_n\}_{n \in \mathbb{N}} \cup \{v\}$, $|\mathsf{e}_D(r_0, r_n)| = \aleph_0$ $(n \in \mathbb{N}^+)$, $|\mathsf{e}_D(r_n, v)| = 1$ $(n \in \mathbb{N}^+)$, $|\mathsf{e}_D(v, r_0)| = \aleph_0$ and $\mathcal{B}_n = (\{r_n\}, \varnothing)$ $(n \in \mathbb{N})$ (see figure 5). We show that the system above does not violate condition (1). Let $\varnothing \neq X \subseteq V$ be arbitrary. Assume first that $r_0 \notin X$. If $r_n \in X$, for some $n \in \mathbb{N}^+$, then $\varrho_{D \setminus\!\!\setminus \mathcal{B}}(X) = \aleph_0$; if not, then $X = \{v\}$ and $\varrho_{D \setminus\!\!\setminus \mathcal{B}}(X) = \aleph_0$ again. Assume $r_0 \in X$. If $v \notin X$, then $\varrho_{D \setminus\!\!\setminus \mathcal{B}}(X) = \aleph_0$. If $v \in X$, then each element of $\{r_n\}_{n \in \mathbb{N}^+} \setminus X$ has one outgoing edge to X, then there is equality in condition (1). Otherwise, there obviously is no system of edge-disjoint paths $\{P_n\}_{n \in \mathbb{N}}$ such that P_n goes from r_n to v, and thus we can not extend the branchings $\mathcal{B}_n$ to edge-disjoint spanning branchings.

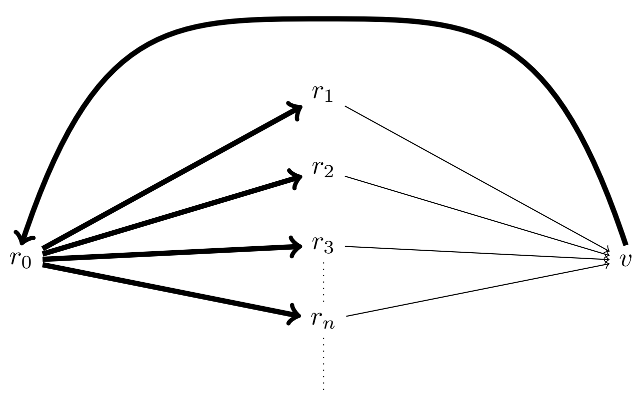

Figure 5: The counterexample. (Thick arrows stand for countably infinite parallel edges.)

Conjecture 10. *Assume that $D = (V, A)$ is a digraph, κ is an infinite cardinal and $\mathcal{B}_i = (V_i, A_i)$ $(i < \kappa)$ are edge-disjoint branchings in D. Let $D \setminus\!\!\setminus \mathcal{B} = (V, A \setminus \cup_{i<\kappa} A_i)$. Suppose $D \setminus\!\!\setminus \mathcal{B}$ does not contain forward-infinite paths, and for all $v \in V$ there is a system of edge-disjoint paths $\{P_i\}_{i<\kappa}$ in $D \setminus\!\!\setminus \mathcal{B}$ such that P_i goes from V_i to v. Then the branchings $\mathcal{B}_i$ can be extended to edge-disjoint spanning branchings of D without changing their root sets.*

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