PROOF OF NASH-WILLIAMS' INTERSECTION CONJECTURE FOR COUNTABLE MATROIDS

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ABSTRACT. We prove that if M and N are finitary matroids on a common countable edge set E then they admit a common independent set I such that there is a bipartition $E = E_M \cup E_N$ for which $I \cap E_M$ spans E_M in M and $I \cap E_N$ spans E_N in N. It answers positively the Matroid Intersection Conjecture of Nash-Williams in the countable case.

1. INTRODUCTION

The Matroid Intersection Conjecture of Nash-Williams [1] has been one of the most important open problem in infinite matroid theory for decades. It contains as a special case the generalization of Menger's theorem to infinite graphs conjectured by Erdős and proved by Aharoni and Berger (see [2] and [3]). The Matroid Intersection Conjecture is a generalization of the Matroid Intersection Theorem of Edmonds [4] to infinite matroids based on the complementary slackness conditions (cardinality is usually an overly rough measure to obtain deep results in infinite combinatorics).

Conjecture 1.1 (Matroid Intersection Conjecture by Nash-Williams, [1]). *If M and N are (potentially infinite) matroids on the same edge set E, then they admit a common independent set I for which there is a bipartition $E = E_M \cup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N.*

A "potentially infinite matroid" originally meant an $(E, \mathcal{I})$ with $\mathcal{I} \subseteq \mathcal{P}(E)$ where:

(i) $\mathcal{I} \neq \varnothing$;
(ii) $\mathcal{I}$ is downward closed;
(iii) For every finite $I, J \in \mathcal{I}$ with $|I| < |J|$, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(iv) If every finite subset of an $X \subseteq E$ is in $\mathcal{I}$, then $X \in \mathcal{I}$.

Matroids satisfying the axioms above are called nowadays *finitary* and they form a proper subclass of matroids. Adapting the terminology introduced by Bowler and Carmesin in [9], we say that the matroid pair $\{M, N\}$ (where $E(M) = E(N)$) has the *Intersection property* if they admit a common independent set demanded by Conjecture 1.1. The first (and for a long time the only) partial result on Conjecture 1.1 was due to Aharoni and Ziv:

Theorem 1.2 (Aharoni and Ziv, [1]). *Let M and N be finitary matroids on the same countable edge set and assume that M is the direct sum of matroids of finite rank. Then $\{M, N\}$ has the Intersection property.*

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Our main result is to omit completely the extra assumption about M:

Theorem 1.3. *Let M and N be finitary matroids on the same countable edge set. Then $\{M, N\}$ has the Intersection property.*

Finitary matroids were not considered as an entirely satisfying infinite generalisation of matroids because they fail to capture a key phenomenon of the finite theory, the duality. Indeed, the class of finitary matroids is not closed under taking duals, namely the set of subsets of E avoiding some $\subseteq$-maximal element of $\mathcal{I}$ does not necessarily satisfy axiom (iv). Rado asked in 1966 if there is a reasonable notion of infinite matroids admitting duality and minors. Among other attempts Higgs introduced [5] a class of structures he called "B-matroids". Oxley gave an axiomatization of B-matroids and showed that they are the largest class of structures satisfying axioms (i)-(iii) and closed under taking duals and minors (see [6] and [7]). Despite these discoveries of Higgs and Oxley, the systematic investigation of infinite matroids started only around 2010 when Bruhn, Diestel, Kriesell, Pendavingh, Wollan found a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms for finite mastoids and showed that several well-known facts of the theory of finite matroids are preserved (see [8]).

An $M = (E, \mathcal{I})$ is a B-matroid (or simply matroid) if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

(1) $\mathcal{I} \neq \varnothing$;
(2) $\mathcal{I}$ is downward closed;
(3) For every $I, J \in \mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ but I is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(4) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

This more general matroid concept gave a broader interpretation for Conjecture 1.1. Bowler and Carmesin showed in [9] that several important conjectures in infinite matroid theory are equivalent with Conjecture 1.1 and gave a simpler proof for a slightly more general form of Theorem 1.2.

To state a more general form of our main result Theorem 1.3 together with a couple of side results, we recall some notions. A matroid is called *finitary* if all of its circuits are finite (equivalently: if satisfies (iv)). The *finitarization* of M is a matroid on the same edge set whose circuits are exactly the finite circuits of M. A matroid M is *nearly finitary* if every base of M can be extended to a base of its finitarization by adding finitely many edges. A matroid is *(nearly) cofinitary* if its dual is (nearly) finitary. The *cofinitarization* of M is the dual of the finitarization of M^*.

Theorem 1.4. *If M and N are matroids on a common countable edge set where each of them is either nearly finitary or nearly cofinitary, then $\{M, N\}$ has the Intersection property.*

For matroids M and N on the same edge set E, $\mathsf{cond}(M, N)$ stands for the condition "for every $W \subseteq E$ for which there is a base of $M \upharpoonright W$ independent in $N.W$, there exists a base of $N.W$ which is independent in $M \upharpoonright W$". The next theorem says that $\mathsf{cond}(M, N)$ is a necessary and sufficient condition for the existence of a set which is independent in M and spanning in N.

Theorem 1.5. *Let M and N be matroids on a common countable edge set such that each of them is either nearly finitary or nearly cofinitary. Then there is a base of N which is independent in M if and only if* $\mathsf{cond}(M, N)$ *holds.*

Looking for an M-independent N-base can be rephrased as searching for an N-base contained in an M-base. It seems natural to ask about a characterisation for having a common base.

Theorem 1.6. *Let M and N be matroids on a common countable edge set such that each of them is either finitary or cofinitary. Then M and N have a common base if and only if* $\mathsf{cond}(M, N) \wedge \mathsf{cond}(N, M)$ *holds.*

Maybe surprisingly, the generalization of Theorem 1.6 for arbitrary countable matroids is consistently false (take U and U^* from Theorem 5.1 of [12]). In contrast to our other results, we do not even know if "finitary or cofinitary" can be relaxed to "nearly finitary or nearly cofinitary" in Theorem 1.6.

It is worth mentioning that if M is finitary and N is cofinitary with $E(M) = E(N)$, then $\{M, N\}$ has the Intersection property regardless of the size $|E(M)|$. Indeed, by applying Zorn's lemma we take a maximal $X \subseteq E$ which admits a bipartition $X = I \cup J$ where I is independent in M and J is independent in N^*. (Since being finitary means having only finite circuts, the conditions of Zorn's lemma hold by an easy compactness argument.) The desired common independent set is I and the witnessing bipartition $E = E_M \cup E_N$ can be obtained by using the method developed by Edmonds and Fulkerson in [11]. This was proven by Aigner-Horev et al. in a more general form:

Theorem 1.7 (Aigner-Horev, Carmesin and Frölich; Theorem 1.5 in [3])**.** *If M is a nearly finitary and N is a nearly cofinitary matroid on a common edge set, then $\{M, N\}$ has the Intersection property.*

The paper is structured as follows. After introducing a few notation in the next section we recall the augmenting path method in Edmonds' proof of the Matroid Intersection Theorem in Section 3 and analyse the changes of the auxiliary digraph after an augmentation. In Section 4 we remind the so called "wave" technique developed by Aharoni and prove some properties of waves. We show in Section 5 that the restriction of Theorem 1.5 to finitary matroids implies all the theorems we are intended to prove and from that point we focus only on this theorem. In Section 6 we investigate feasible sets, i.e., common independent sets I of M and N satisfying $\mathsf{cond}(M/I, N/I)$. The intended meaning of "feasible" is being extendable to an M-independent base of N. The main result is proved in Section 7 and its core is Lemma 7.1 which enables us to find a feasible extension of a given feasible set which spans in N a prescribed edge.

2. Notation and basic facts

In this section we introduce some notation and recall some basic facts about matroids that we will use later without further explanation. For more details we refer to [10].

A pair $M=(E,\mathcal{I})$ is a *matroid* if $\mathcal{I}\subseteq\mathcal{P}(E)$ satisfies the axioms (1)-(4). The sets in $\mathcal{I}$ are called *independent* while the sets in $\mathcal{P}(E)\setminus\mathcal{I}$ are *dependent*. An $e\in E$ is a *loop* if $\{e\}$ is dependent. If E is finite, then (1)-(3) are equivalent to the usual axiomisation of matroids in terms of independent sets (while (4) is automatically true). The maximal independent sets are called *bases* and the minimal dependent sets are called *circuits*. Every dependent set contains a circuit (which fact is not obvious if E is infinite). If C_1, C_2 are circuits with $e\in C_1\setminus C_2$ and $f\in C_1\cap C_2$, then there is a circuit C_3 with $e\in C_3\subseteq C_1\cup C_2-f$. We say that C_3 is obtained by *strong circuit elimination* from C_1 and C_2 keeping e and removing f. The *dual* of a matroid M is the matroid M^* with $E(M^*)=E(M)$ whose bases are the complements of the bases of M. For an $X\subseteq E$, $\boldsymbol{M\upharpoonright X}:=(X,\mathcal{I}\cap\mathcal{P}(X))$ is a matroid and it is called the *restriction* of M to X. We write $\boldsymbol{M-X}$ for $M\upharpoonright(E\setminus X)$ and call it the minor obtained by the *deletion* of X. The *contraction* of X in M and the contraction of M onto X are $\boldsymbol{M/X}:=(M^*\upharpoonright X)^*$ and $\boldsymbol{M.X}:=M/(E\setminus X)$ respectively. Contraction and deletion commute, i.e., for disjoint $X,Y\subseteq E$, we have $(M/X)-Y=(M-Y)/X$. Matroids of this form are the *minors* of M. If I is independent in M but $I+e$ is dependent for some $e\in E\setminus I$ then there is a unique circuit $\boldsymbol{C_M(e,I)}$ of M through e contained in $I+e$. We say $X\subseteq E$ *spans* $e\in E$ in matroid M if either $e\in X$ or there exists a circuit $C\ni e$ with $C-e\subseteq X$. We denote the set of edges spanned by X in M by $\boldsymbol{\mathsf{span}_M(X)}$. An $S\subseteq E$ is *spanning* in M if $\mathsf{span}_M(S)=E$. Let us define $\boldsymbol{\|X\|}$ to be $|X|$ if it is finite and ∞ otherwise. Let B and B_X be a base of M and $M\upharpoonright X$ respectively with $B_X\subseteq B$. Then $\|B\setminus B_X\|$ does not depend on the choice of B and B_X and called the *corank* $\boldsymbol{c_M(X)}$ of X in M.

3. Augmenting paths

The Matroid Intersection Theorem states (using our terminology) that every pair of matroids on the same finite edge set has the Intersection property. It is a fundamental tool in combinatorial optimization and has a great importance since it has been discovered by Edmonds [4]. The polynomial algorithm in Edmonds' proof finds a maximal sized common independent set together with a bipartition witnessing optimality. It improves a common independent set iteratively via augmenting paths taken in an auxiliary digraph.

In the infinite case these augmenting paths are working in the same way and will play an important role in our proof. However they are not sufficient alone to prove our main result. Indeed, applying augmenting paths recursively yields a sequence of common independent sets where a reasonable limit object cannot be guaranteed in general. In this subsection we introduce our terminology about augmenting paths and prove some properties which were irrelevant for Edmonds' proof but are crucial for our arguments.

Let N and M be fixed arbitrary matroids on the same edge set E. For a common independent set I, let $\boldsymbol{D(I,N,M)}$ be a digraph on E with the following arcs. For $e\in I$ and $f\in E\setminus I$, $ef\in D(I,N,M)$ if $f\in\mathsf{span}_N(I)$ with $e\in C_N(f,I)$ and $fe\in D(I,N,M)$ if $f\in\mathsf{span}_M(I)$ with $e\in C_M(f,I)$. Note that $D(I,M,N)$ is obtained from $D(I,N,M)$ by reversing all the arcs. An *augmenting path* with respect to the triple (I,N,M) is a $\subseteq$-minimal $P\subseteq E$ of odd size admitting a linear ordering $P=\{x_0,\dots,x_{2n}\}$, for which

(1) $x_0\in E\setminus\mathsf{span}_N(I)$,

(2) $x_{2n} \in E \setminus \mathsf{span}_M(I)$,
(3) $x_k x_{k+1} \in D(I, N, M)$ for $k < 2n$.

Observe that each x_k with $0 < k < 2n$ is spanned by I in both matroids. Furthermore, by the minimality of P there cannot be $k + 1 < \ell$ with $x_k x_\ell \in D(I, N, M)$ (i.e., there are no jumping arcs). Therefore the linear order witnessing that P is an augmenting path for (I, N, M) is unique. Clearly augmenting paths for (I, N, M) and (I, M, N) are the same (the witnessing orderings are the reverse of each other) thus being augmenting path for I and $\{M, N\}$ is well-defined. If there is no augmenting path for I then the set E_M of elements reachable from $E \setminus \mathsf{span}_N(I)$ in $D(I, N, M)$ together with $E_N := E \setminus E_M$ witnessing the Intersection property of $\{N, M\}$.

It allows us to give an alternative characterization of the common independent sets in Conjecture 1.1. An element I of a set family $\mathcal{F}$ is called *strongly maximal* in $\mathcal{F}$ if $\|J \setminus I\| \leq \|I \setminus J\|$ for every $J \in \mathcal{F}$. It is known that every maximal independent set of a matroid is a strongly maximal independent set as well (see Lemma 3.7 in [8]). On the one hand, if I is as in Conjecture 1.1, then its strong maximality among the common independent sets is ensured by the properties of the bipartition $E = E_N \cup E_M$, i.e., the fact that I_M is a strongly maximal independent set of $M \upharpoonright E_M$ as well as I_N in $N \upharpoonright E_N$. On the other hand, an augmenting path P has always one more element in $E \setminus I$ than in I, furthermore, $I \vartriangle P$ is known to be a common independent set. Hence if I is a strongly maximal common independent set, then there cannot exist any augmenting path which yields to a desired bipartition. Therefore having the Intersection property is equivalent to admitting a strongly maximal common independent set.

Let an augmenting path $P = \{x_0, \dots, x_{2n}\}$ for (I, N, M) be fixed.

Lemma 3.1. *If P contains neither e nor any of its out-neighbours with respect to $D(I, N, M)$, then $ef \in D(I \vartriangle P, N, M)$ whenever $ef \in D(I, N, M)$.*

Proof. For an $e \in E \setminus I$, its out-neighbours are $C_M(e, I) - e$ (or $\varnothing$ which case is irrelevant). By assumption $P \cap C_M(e, I) = \varnothing$ and therefore $C_M(e, I) = C_M(e, I \vartriangle P)$. This means by definition that e has the same out-neighbours in $D(I, N, M)$ and $D(I \vartriangle P, N, M)$.

Assume now that $e \in I$ and $ef \in D(I, N, M)$ (i.e., $e \in C_N(f, I)$) for some f. For $k \leq n$, let us denote $I + x_0 - x_1 + x_2 - \dots - x_{2k-1} + x_{2k}$ by I_k. Observe that $I_n = I \vartriangle P$. We show by induction on k that I_k is N-independent and $e \in C_N(f, I_k)$. Since $I + x_0$ is N-independent by definition and $x_0 \neq f$ by assumption, we obtain $C_N(f, I) = C_N(f, I_0)$. Suppose that we already know the statement for some $k < n$. We have $C_N(x_{2k+2}, I_k) = C_N(x_{2k+2}, I) \ni x_{2k+1}$ because there is no jumping arc in the augmenting path. It follows that I_{k+1} is N-independent. If $x_{2k+1} \notin C_N(f, I_k)$ then $C_N(f, I_k) = C_N(f, I_{k+1})$ and the induction step is done. Suppose that $x_{2k+1} \in C_N(f, I_k)$. Note that $e \notin C_N(x_{2k+2}, I)$ since otherwise P would contain the out-neighbour x_{2k+2} of e in $D(I, N, M)$. We apply strong circuit elimination with $C_N(f, I_k)$ and $C_N(x_{2k+2}, I_k)$ keeping e and removing x_{2k+1}. The resulting circuit $C \ni e$ can have at most one element out of I_{k+1}, namely f. Since I_{k+1} is N-independent, there must be at least one such an element and therefore $C = C_N(f, I_{k+1})$. □

Corollary 3.2. $\mathsf{span}_N(I \vartriangle P) = \mathsf{span}_N(I + x_0)$ *and* $\mathsf{span}_M(I \vartriangle P) = \mathsf{span}_M(I + x_{2n})$.

Proof. By symmetry it is enough to prove the first equality. In the proof of Lemma 3.1, I_{k+1} is obtained from I_k by replacing $x_{2k+1} \in I_k$ by x_{2k+2} for which $x_{2k+1} \in C_N(x_{2k+2}, I_k)$ thus $\mathsf{span}_N(I_k) = \mathsf{span}_N(I_{k+1})$. Since $I_0 = I + x_0$ and $I_n = I \triangle P$ we are done by induction. □

Observation 3.3. *If $ef \in D(I, N, M)$ and $J \supseteq I$ is a common independent set of N and M with $\{e, f\} \cap J = \{e, f\} \cap I$, then $ef \in D(J, N, M)$ (the same circuit is the witness).*

4. Waves

Waves were introduced by Aharoni to solve problems in infinite matching theory. These techniques turned out to be useful in the proof of the Erdős-Menger Conjecture by Aharoni and Berger [2] and in the already mentioned result [1] about the Matroid Intersection Conjecture. Let M and N be arbitrary matroids on the same edge set E. An (M, N)-*wave* is a $W \subseteq E$ such that there is a base of $M \upharpoonright W$ which is independent in $N.W$. If (M, N) is clear from the context we write simply wave. A set L of M-loops is a wave witnessed by $\varnothing$. We call such a wave *trivial*.

Proposition 4.1. *The union of arbitrary many waves is a wave.*

Proof. Suppose that W_β is a wave for $\beta < \kappa$ and let $W_{<\alpha} := \bigcup_{\beta<\alpha} W_\beta$ for $\alpha \leq \kappa$. We fix a base $B_\beta \subseteq W_\beta$ of $M \upharpoonright W_\beta$ which is independent in $N.W_\beta$. Let us define $B_{<\alpha}$ by transfinite recursion for $\alpha \leq \kappa$ as follows.

$$B_{<\alpha} := \begin{cases} \varnothing & \text{if } \alpha = 0 \\ B_{<\beta} \cup (B_\beta \setminus W_{<\beta}) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} B_{<\beta} & \text{if } \alpha \text{ is limit ordinal.} \end{cases}$$

First we show by transfinite induction that $B_{<\alpha}$ is spanning in $M \upharpoonright W_{<\alpha}$. For $\alpha = 0$ it is trivial. For a limit α it follows directly from the induction hypothesis. If $\alpha = \beta + 1$, then by the choice of B_β, the set $B_\beta \setminus W_{<\beta}$ spans $W_{<\beta+1} \setminus W_{<\beta}$ in $M/W_{<\beta}$. Since $W_{<\beta}$ is spanned by $B_{<\beta}$ in M by induction, it follows that $W_{<\beta+1}$ is spanned by $B_{<\beta+1}$ in M.

The independence of $B_{<\alpha}$ in $N.W_{<\alpha}$ can be reformulated as "$W_{<\alpha} \setminus B_{<\alpha}$ is spanning in $N^* \upharpoonright W_{<\alpha}$", which can be proved the same way as above. □

By Proposition 4.1 there exists a $\subseteq$-largest (M, N)-wave that we denote by $\boldsymbol{W(M, N)}$. Note that if $W(M, N)$ is not witnessing the violation of $\mathsf{cond}(M, N)$ (see the definition right after Theorem 1.4) then there is no such a witness, i.e., $\mathsf{cond}(M, N)$ holds.

Observation 4.2. *If W_0 is an (M, N)-wave and W_1 is an $(M/W_0, N - W_0)$-wave, then $W_0 \cup W_1$ is an (M, N)-wave.*

Corollary 4.3. *For $W := W(M, N)$, the largest $(M/W, N - W)$-wave is $\varnothing$.*

Observation 4.4. *If $\mathsf{cond}(M, N)$ holds and L consists of M-loops, then $L \subseteq \mathsf{span}_N(E \setminus L)$ since otherwise wave L would violate $\mathsf{cond}(M, N)$).*

Corollary 4.5. *Assume that $\mathsf{cond}(M, N)$ holds, $X \subseteq E$ and $L \subseteq X$ consists of M-loops. Then any base B of $(N.X) - L$ is a base of $N.X$.*

Proof. For a base B' of $N - X$, the set $B \cup B'$ spans $E \setminus L$ and hence by Observation 4.4 spans the whole E as well. □

Let us write $\mathsf{cond}^+(M, N)$ for the condition that $\mathsf{cond}(M, N)$ holds and $W(M, N)$ is trivial (i.e., consists of M-loops).

Lemma 4.6. *$\mathsf{cond}^+(M, N)$ implies that whenever W is a $(M/e, N/e)$-wave for some $e \in E$ witnessed by $B \subseteq W$, then B is a common base of $M \upharpoonright W$ and $N.W$.*

Proof. Let W be an $(M/e, N/e)$-wave. Note that $(N/e).W = N.W$ by definition. Pick an $B \subseteq W$ which is an $N.W$-independent base of $(M/e) \upharpoonright W$. We may assume that $e \in \mathsf{span}_M(W)$ and e is not an M-loop. Indeed, otherwise $(M/e) \upharpoonright W = M \upharpoonright W$ holds and hence by $\mathsf{cond}^+(M, N)$ we may conclude that W is trivial and $B = \varnothing$ is a desired common base.

Then B is not a base in $M \upharpoonright W$ but "almost", namely $c_{M \upharpoonright W}(B) = 1$. We apply the augmenting path method with $B, M \upharpoonright W, N.W$. The augmentation cannot be successful. Indeed, if P were an augmenting path then $B \bigtriangleup P$ would show that W is a non-trivial (M, N)-wave. Thus we get a bipartition $W = W_0 \cup W_1$ where $B \cap W_0$ spans W_0 in M and $B \cap W_1$ spans W_1 in $N.W$. Observe that W_0 is an (M, N)-wave and hence it must be trivial. By applying Corollary 4.5 with $X = W$ and $L = W_0$, we may conclude that B is a base of $N.W$ (and of $M \upharpoonright W$ by definition). □

5. Reductions

The first reduction (Corollary 5.4) will connect Theorems 1.4 and 1.5 even in a more general form. This was already discovered by Aharoni and Ziv in [1].

Proposition 5.1. *Let M and N be matroids on the common edge set E such that $\{M, N\}$ has the Intersection property. Then $\mathsf{cond}(M, N)$ is equivalent with the existence of an M-independent base of N.*

Proof. The condition $\mathsf{cond}(M, N)$ is clearly necessary even without any further assumption. To show its sufficiency, let $I = I_M \cup I_N$ and $E = E_M \cup E_N$ as in Conjecture 1.1. Then I_M is an $N.E_M$-independent base of $M \upharpoonright E_M$ and I_N is an $M.E_N$-independent base of $N \upharpoonright E_N$. Therefore E_M is a wave and by $\mathsf{cond}(M, N)$ we can pick a J which is a base of $N.E_M$ and independent in M. Then $B := I_N \cup J$ is a base of N and it is also independent in M because I_N is independent in $M.E_N$. □

Observation 5.2. *The matroid classes: finitary, cofinitary, nearly finitary, nearly cofinitary are closed under taking minors. Furthermore, if κ is a cardinal and class $\mathcal{C}$ closed under taking minors, then so is the subclass $\{M \in \mathcal{C} : |E(M)| < \kappa\}$.*

Proposition 5.3. *Assume that $\mathcal{C}$ is a class of matroids closed under taking minors such that for every $(M, N) \in \mathcal{C} \times \mathcal{C}$ with $E(M) = E(N)$, $\mathsf{cond}(M, N)$ implies the existence of a base of N which is independent in M. Then every pair $\{M, N\}$ from $\mathcal{C}$ with $E(M) = E(N)$ has the Intersection property.*

Proof. Let $E_M := W(M, N)$ and let I_M be a base of $M \upharpoonright E_M$ which is independent in $N.E_M$, i.e., I_M is a witness that E_M is a wave. Then $W(M/E_M, N - E_M) = \varnothing$ by Corollary

4.3, in particular $\mathsf{cond}(M/E_M, N - E_M)$ holds. Since $\mathcal{C}$ is closed under taking minors, we have $M/E_M, N - E_M \in \mathcal{C}$ and therefore by assumption we can find a base I_N of $N - E_M$ which is independent in M/E_M. □

Corollary 5.4. *Let $\mathcal{C}$ be a class of matroids which is closed under taking minors. The following are equivalent:*

(1) For every $M, N \in \mathcal{C}$ with $E(M) = E(N)$, $\{M, N\}$ has the Intersection property.
(2) For every $M, N \in \mathcal{C}$ with $E(M) = E(N)$, there is a base of N which is independent in M if and only if $\mathsf{cond}(M, N)$.

Our next goal is to show that the Matroid Intersection Conjecture 1.1 for nearly finitary and nearly cofinitary matroids can be reduced to finitary and cofinitary ones even if the matroids are not countable, more precisely:

Proposition 5.5. *For $i \in \{0, 1\}$, let M_i be a nearly finitary (nearly cofinitary) matroid on E and let M_i' be its (co)finitarization. If $\{M_0', M_1'\}$ has the Intersection property then so does $\{M_0, M_1\}$.*

Proof. Suppose first that the M_i are both nearly finitary. Let I' be a common independent set of M_0' and M_1' and let $E = E_0 \cup E_1$ be a bipartition as in Conjecture 1.1. By definition, for $I_i' := I' \cap E_i$ we have $c_{M_i' \restriction E_i}(I_i') = 0$. Observe that $c_{M_i \restriction E_i}(X) \leq c_{M_i' \restriction E_i}(X)$ for every $X \subseteq E$ because every circuit of $M_i' \restriction E_i$ is a circuit of $M_i \restriction E_i$. From the definition of "nearly finitary" follows directly that we can delete finitely many elements of I' to obtain a common independent set I of M_0 and M_1. Then for $I_i := I \cap E_i$ we have $c_{M_i' \restriction E_i}(I_i) < \infty$ and hence by the observation above $c_{M_i \restriction E_i}(I_i) < \infty$ as well.

We use the augmenting path method with M_0, M_1 and I. If there is no augmenting path then I is as desired and we are done. Otherwise we take an augmenting path P. Since P has one more elements in $E \setminus I$ than in I, for $J := I \triangle P$ we have $|J \setminus I| = |I \setminus J| + 1 < \infty$. Thus $\sum_{i=0,1} |J_i \setminus I_i| = 1 + \sum_{i=0,1} |I_i \setminus J_i|$ where $J_i := J \cap E_i$. Therefore $\sum_{i=0,1} c_{M_i \restriction E_i}(J_i) < \sum_{i=0,1} c_{M_i \restriction E_i}(I_i)$. It follows that after finitely many iterative application of augmenting paths we must obtain the desired strongly maximal common independent set.

If say M_0 is nearly cofinitary then the independence in M_0' implies the independence in M_0. Although the inequality $c_{M_0 \restriction E_0}(X) \leq c_{M_0' \restriction E_0}(X)$ for $X \subseteq E$ is not true in general, it follows from the definition of "nearly cofinitary" directly that for every $X \subseteq E_0 :\ c_{M_0' \restriction E_0}(X) < \infty$ implies $c_{M_0 \restriction E_0}(X) < \infty$. Based on this implication the proof of the nearly finitary case above can be adapted for the nearly cofinitary and mixed cases. □

Observation 5.6 (Bowler and Carmesin, [9])**.** *If $\{M, N\}$ has the Intersection property witnessed by I and $E = E_M \cup E_N$, then $\{M^*, N^*\}$ has it as well showed by $E \setminus I, E = E_{M^*} \cup E_{N^*}$ where $E_{M^*} = E_N$ and $E_{N^*} = E_M$.*

We will prove in the rest of the paper the restriction of Theorem 1.5 to finitary matroids. All of our results follow from it. Indeed, it implies Theorem 1.4 for finitary matroids (see Corollary 5.4 and Observation 5.2). Then the generalization to nearly finitary matroids can be obtained by Proposition 5.5 from which the nearly cofinitary case follows by Observation 5.6. The nearly finitary-nearly cofinitary case is solved in Theorem 1.7. Finally, from

Theorem 1.4 we get Theorem 1.5 by Corollary 5.4 from which Theorem 1.6 follows by applying the following result.

Theorem 5.7 (Corollary 1.4 in [12]). *Let M_i be a finitary or cofinitary matroid on the edge set E for $i \in \{0,1\}$. If there are bases B_i, B_i' of M_i such that $B_0 \subseteq B_1$ and $B_1' \subseteq B_0'$, then M_0 and M_1 share some base.*

6. Feasible sets

Let M and N be some fixed matroids on the same edge set E. An $I \subseteq E$ is *feasible* (with respect to (M,N)) if I is a common independent set of M and N such that $\mathsf{cond}(M/I, N/I)$ holds. Note that $\mathsf{cond}(M,N)$ says that $\varnothing$ is feasible, moreover, if Theorem 1.5 is true, then exactly the feasible sets can be extended to a base of N which is independent in M. A feasible I is called *nice* if $\mathsf{cond}^+(M/I, N/I)$ holds (see the definition right before Lemma 4.6).

Observation 6.1. *If I_0 is a common independent set and I_1 is feasible with respect to $(M/I_0, N/I_0)$, then $I_0 \cup I_1$ is feasible with respect to (M,N). If in addition I_1 is a nice feasible set with respect to $(M/I_0, N/I_0)$, then so is $I_0 \cup I_1$ for (M,N).*

Lemma 6.2. *If B is a common base of $M \upharpoonright W$ and $N.W$ for $W := W(M,N)$, then B is a nice feasible set.*

Proof. Let $W' := W(M/B, N/B)$. First we show that $W' = W \setminus B$ and it consists of M/B-loops. On the one hand, B is spanning in $M \upharpoonright W$ thus $W \setminus B$ consists of M/B-loops which gives $W' \supseteq W \setminus B$. On the other hand, let J be a witness that W' is an $(M/B, N/B)$-wave. Then $B \cup J$ ensures that $W \cup W'$ is an (M,N)-wave. Therefore $W' \subseteq W$ which yields to $W' \subseteq W \setminus B$. Thus $W' = W \setminus B$ consists of M/B-loops as promised. It remains to show that $\mathsf{cond}(M/B, N/B)$ holds. From the fact that B is a base of $N.W$ we can conclude that $\varnothing$ is a base of $N.(W \setminus B)$ which completes the proof. ☐

Remark 6.3. One may observe that if each of M and N are either finitary or cofinitary then $\mathsf{cond}(M,N)$ implies via Theorem 5.7 that for every wave W there exists a common base B of $M \upharpoonright W$ and $N.W$. For self readability reasons we are not intended to use this in the proof of the main result.

Lemma 6.4. *If I is a nice feasible set and P is an augmenting path for it, then $I \bigtriangleup P$ can be extended to a nice feasible set.*

Proof. It is enough to find a common base B of $M/(I \bigtriangleup P) \upharpoonright W$ and $(N/(I \bigtriangleup P)).W$ where $W := W(M/(I \bigtriangleup P), N/(I \bigtriangleup P))$. Indeed, then we are done by applying Lemma 6.2 and Observation 6.1. Let e be the unique element of $P \setminus \mathsf{span}_M(I)$. Corollary 3.2 ensures that $I+e$ and $I \bigtriangleup P$ span each other in M therefore $M/(I+e) \upharpoonright X = M/(I \bigtriangleup P) \upharpoonright X$ whenever $X \subseteq E \setminus (I \cup P)$. For such an X we also have $N.X = (N/(I+e)).X = (N/(I \bigtriangleup P)).X$. In particular the wave subsets of $E \setminus (I \cup P)$ and the associated minors of M and N are identical in $(M/(I \bigtriangleup P), N/(I \bigtriangleup P))$ and in $(M/(I+e), N/(I+e))$. Let W' be the union of all these common waves. On the one hand, each $e \in I \cap P$ is a common loop of $M/(I \bigtriangleup P)$ and $N/(I \bigtriangleup P)$ by Corollary 3.2. Hence $W = W' \cup (I \cap P)$, furthermore, a common base of $M/(I \vartriangle P) \upharpoonright W'$ and $(N/(I \vartriangle P)).W'$ is automatically a common base of $M/(I \vartriangle P) \upharpoonright W$ and $(N/(I \vartriangle P)).W$ as well. On the other hand, by applying Lemma 4.6 with M/I and N/I and e, there exists a common base B of $M/(I+e) \upharpoonright W'$ and $(N/(I+e)).W'$. This B is a common base of $M/(I \vartriangle P) \upharpoonright W'$ and $(N/(I \vartriangle P)).W'$ since $W' \subseteq E \setminus (I \cup P)$. □

7. The proof of the main result

We may assume without loss of generality in the proof of Theorem 1.5 that $\mathsf{cond}^+(M,N)$ holds. Indeed, otherwise we consider $(M/W, N-W)$ instead for $W := W(M,N)$. By Corollary 4.3, $W(M/W, N-W) = \varnothing$, thus in particular $\mathsf{cond}^+(M/W, N-W)$ holds. Finally the union of an M/W-independent base of $N-W$ and an M-independent base of $N.W$ (exists by $\mathsf{cond}(M,N)$) is a desired M-independent base of N.

Lemma 7.1. *If M and N are finitary matroids on the common countable edge set E such that $\mathsf{cond}^+(M,N)$ holds, then for every $e \in E$, there exists a nice feasible I with $e \in \mathsf{span}_N(I)$.*

Let us fix an enumeration $\{e_n : n \in \mathbb{N}\}$ of E and take a well-order $\prec$ on E according to it. Theorem 1.5 for finitary matroids follows from Lemma 7.1 by a straightforward recursion. Indeed, we build an $\subseteq$-increasing sequence (I_n) of nice feasible sets starting with $I_0 := \varnothing$ in such a way that $e_n \in \mathsf{span}_N(I_{n+1})$. If I_n is already defined and $e_n \notin \mathsf{span}_N(I_n)$, then we apply Lemma 7.1 with $(M/I_n, N/I_n)$ and e_n and take the union of the resulting J with I_n to obtain I_{n+1} (see Observation 6.1). Using that M and N are finitary, we conclude that $\bigcup_{n=0}^{\infty} I_n$ is a base of N which is independent in M.

proof of Lemma 7.1. It is enough to build a sequence (I_n) of nice feasible sets such that $\mathsf{span}_N(I_n)$ is monotone $\subseteq$-increasing in n and $\bigcup_{n=0}^{\infty} \mathsf{span}_N(I_n) = E$. We start with $I_0 = \varnothing$ and apply an augmenting path and add some new edges at each step. Corollary 3.2 ensures that $\mathsf{span}_N(I_n)$ is monotone $\subseteq$-increasing. Suppose I_n is already defined. Assume first that there is no augmenting path for I_n. Then there is a bipartition $E = E_M \cup E_N$ witnessing with I_n the Intersection property of $\{M,N\}$. By definition, E_M is a wave and it must be trivial by $\mathsf{cond}^+(M,N)$. Therefore $I_n \subseteq E_N$ and it is spanning in N by Observation 4.4. Hence $B := I_n$ is a base of N which is independent in M.

We may assume that there exists some augmenting path for I_n. Consider the $\prec$-smallest $e \in E \setminus \mathsf{span}_N(I_n)$ for which there is an augmenting path P_n such that $e \in \mathsf{span}_N(I_n \vartriangle P_n)$. Lemma 6.4 ensures that we can extend $I_n \vartriangle P_n$ to a nice feasible set I_{n+1}. The recursion is done.

Suppose for a contradiction that $\boldsymbol{X} := E \setminus \bigcup_{n=0}^{\infty} \mathsf{span}_N(I_n) \neq \varnothing$.

Observation 7.2. *Since N is finitary, Observation 4.4 ensures that there is an edge in X which is not an M-loop.*

For $x \in X$, let $\boldsymbol{E(x,n)}$ be the set of edges that are reachable from x in $\boldsymbol{D_n} := D(I_n, N, M)$ by a directed path. Let $\boldsymbol{n_x}$ be the smallest natural number such that for every $y \in E \setminus X$ with $y \prec x$ we have $y \in \mathsf{span}_N(I_{n_x})$.

Claim 7.3. *For every $x \in X$ and $\ell \geq m \geq n_x$,*

(1) $I_m \cap E(x,m) = I_\ell \cap E(x,m)$,
(2) $C_M(e, I_\ell) = C_M(e, I_m) \subseteq E(x,m)$ *for every* $e \in E(x,m) \setminus I_m$,
(3) $D_m[E(x,m)]$ *is a subdigraph of* $D_\ell[E(x,m)]$,
(4) $E(x,m) \subseteq E(x,\ell)$.

Proof. Suppose that there is an $n \geq n_x$ such that we know already the statement whenever $m, \ell \leq n$. For the induction step it is enough to show that the claim holds for n and $n+1$.

Proposition 7.4. $P_n \cap E(x,n) = \varnothing$.

Proof. A meeting of P_n and $E(x,n)$ would show that there is also an augmenting path in D_n starting at x which is impossible since $x \in X$ and $n \geq n_x$. □

Corollary 7.5. $I_n \cap E(x,n) = (I_n \triangle P_n) \cap E(x,n)$.

Proposition 7.6. $(I_n \triangle P_n) \cap E(x,n) = I_{n+1} \cap E(x,n)$.

Proof. The edges $I_{n+1} \setminus (I_n \triangle P_n)$ are independent in $M/(I_n \triangle P_n)$ but by the definition of D_n for every $e \in E(x,n) \setminus I_n$ we have $E(x,n) \supseteq C_M(e, I_n) = C_M(e, I_n \triangle P_n)$ witnessing that e is an $M/(I_n \triangle P_n)$-loop. □

Corollary 7.7. $I_n \cap E(x,n) = I_{n+1} \cap E(x,n)$ *and for every* $e \in E(x,n) \setminus I_n$ *we have* $C_M(e, I_n) = C_M(e, I_{n+1}) \subseteq E(x,n)$.

Finally, for $e \in E(x,n)$, P_n does not meet e or any of its out-neighbours with respect to D_n because $P \cap E(x,n) = \varnothing$. Hence by applying Lemma 3.1 with e, I_n and D_n (and then Observation 3.3) we may conclude that $ef \in D_{n+1}$ whenever $ef \in D_n$. It follows that $D_n[E(x,n)]$ is a subdigraph of $D_{n+1}[E(x,n)]$ which implies $E(x,n) \subseteq E(x,n+1)$ since reachability is witnessed by the same directed paths. □

Beyond Claim 7.3 we will need the following technical statement.

Proposition 7.8. *Let* I *be an independent set in some fixed finitary matroid. Suppose that there is a circuit* $C \subseteq \mathsf{span}(I)$ *with* $e \in I \cap C$. *Then there is an* $f \in C \setminus I$ *with* $e \in C(f, I)$.

Proof. We apply induction on $|C \setminus I|$. If $C \setminus I$ is a singleton, then its only element is suitable for f since $C(f,I) = C$. Suppose that $|C \setminus I| \geq 2$ and pick a $g \in C \setminus I$. If $e \in C(g, I)$, then $f := g$ is as desired. Otherwise by applying strong circuit elimination with C and $C(g, I)$ keeping e and removing g. The resulting C' satisfies the premisses of the proposition and $C' \setminus I \subsetneq C \setminus I$ holds thus we are done by induction. □

To get the desired contradiction, we show that $\boldsymbol{W} := \bigcup_{x \in X} \bigcup_{n=n_x}^{\infty} E(x,n)$ is a non-trivial wave. Note that property 1 at Claim 7.3 guarantees that for each $e \in W$ either $\{n \in \mathbb{N} : e \in I_n\}$ or its complement is finite. Let J consists of the latter type of edges of W, i.e., that are elements of I_n for every large enough n. Since M and N are finitary, J is a common independent set. By property 2, $W \subseteq \mathsf{span}_M(J)$. We show that J is independent in $N.W$. Suppose for a contradiction that there exists an N-circuit C that meets J but avoids $W \setminus J$. Since J is N-independent and C does not meet $W \setminus J$, we have $C \setminus J = C \setminus W \neq \varnothing$. Let us pick some $e \in C \cap J$. For every large enough n we have $C \cap J = C \cap I_n$ and I_n spans C in N (for the latter we use $X \subseteq W \setminus J$). Applying Proposition 7.8 with I_n, N, C and e tells that $e \in C_N(f, I_n)$ for some $f \in C \setminus W$ whenever n is large enough. Then we can take an $x \in X$ and an $n \geq n_x$ such that $e \in E(x,n) \cap C_N(f, I_n)$ for some $f \in C \setminus W$. Then by definition $f \in E(x,n) \subseteq W$ which contradicts $f \in C \setminus W$. Thus J is indeed independent in $N.W$ and hence W is a wave. Observation 7.2 guarantees that W is non-trivial which contradicts $\mathsf{cond}^+(M,N)$. □

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