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A simple test for comparing regression curves versus one-sided alternatives

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Abstract

In this article we present a simple procedure to test for the null hypothesis of equality of two regression curves versus *one-sided* alternatives in a general nonparametric and heteroscedastic setup. The test is based on the comparison of the sample averages of the estimated residuals in each regression model under the null hypothesis. The test statistic has asymptotic normal distribution. Some simulations and an application to a data set are included.

Key Words: nonparametric regression; comparison of regression curves; nonparametric regression residuals.

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1 Introduction

The comparison of two populations is a common problem in statistics. In a conditional setting, the comparison is usually established between the regression curves, which describe the dependence between the response and the associated covariates. The equality of the regression curves means that the effect of the covariates over the response is the same in both populations.

When the regression curves are included in a parametric family (linear, polynomial, etc.), the problem of comparison of regression curves can be reduced to the comparison of the corresponding parameters. However, in many practical situations no reasonable parametric model can be assumed, and even in that situation the comparison of the regression curves is still an appealing problem (see examples in Härdle and Marron, 1990).

In this article we consider the problem of testing for the equality of two regression curves versus *one-sided* alternatives in a general nonparametric and heteroscedastic setup. More precisely, consider two pairs of variables (X_1, Y_1) and (X_2, Y_2), such that, for $j = 1, 2$, X_j represents a covariate and Y_j is the response or variable of interest. The relationship between the covariate and the response in each population is modeled via a nonparametric regression model of the form

$$Y_j = m_j(X_j) + \varepsilon_j \tag{1}$$

where, for $j = 1, 2$, $m_j(x) = E(Y_j|X_j = x)$ is a smooth regression function, and the regression error verifies $E(\varepsilon_j|X_j = x) = 0$ and $Var(\varepsilon_j|X_j = x) = \sigma_j^2(x)$. We assume that the covariates X_1 and X_2 have common support R_X. The null hypothesis of our testing problem states the equality of the regression curves

$$H_0 : m_1(x) = m_2(x) \text{ for all } \ x \in R_X. \tag{2}$$

In many practical situations, some information can be given about the alternative hypothesis. In this article we focus on a *one-sided alternative hypothesis*, which states that one function is always equal or greater than the other:

$$H_1 : m_1(x) \leq m_2(x) \text{ for all } x \in R_X, \text{ and } m_1 < m_2 \text{ on a set of positive measure.} \tag{3}$$

We propose a very simple testing procedure, which consists in the comparison of the sample averages of the regression residuals estimated nonparametrically under the null hypothesis. See Section 2 for a detailed explanation of the testing procedure.

The problem of testing for the equality of two regression curves versus one-sided alternatives has been considered in the literature by several authors. For instance, the articles by Hall, Huber and Speckman (1997) and Koul and Schick (1997, 2003) work on covariate-matched approaches. Speckman, Chiu, Hewett and Bertelson (2003) consider a test based on signed ranks of residuals under a restrictive version of model (1), with homoscedastic errors and equal design densities. Finally, Neumeyer and Dette (2005) reconsidered the test proposed by Speckman *et al.* (2003) and extended it to completely nonparametric and heteroscedastic models.

A different approach consists of testing the equality of two regression curves against the general alternative $H_0 : m_1(x) \neq m_2(x)$. See Neumeyer and Dette (2003) for a review on this problem. More recently, Pardo-Fernández, Van Keilegom and González-Manteiga (2007) studied a test based on the comparison of the error distributions estimated nonparametrically, which is quite related to the method we will introduce here.

The rest of the paper is organized as follows. Section 2 describes the testing procedure in detail. In Section 3, Theorem 1 states the asymptotic distribution of the test statistic under the null hypothesis, fixed alternatives and local alternatives. Section 4 shows a simulation study. In Section 5, we illustrate the testing procedure with an application to a data set concerning annual expenditures of Dutch households. Finally, the appendix contains the proof of the main result in Section 3.

2 Testing procedure and asymptotic results

Pardo-Fernández *et al.* (2007) proposed a test for the equality of several regression curves versus general alternatives. Their test, which is consistent under general alternatives, is based on the comparison of two empirical estimators of the error distribution in each population (one constructed under the null hypothesis of equal regression curves and the other under the general nonparametric model).

When the alternative hypothesis is of one-sided type, as presented in (3), the test can be simplified, as we will explain in the sequel. Let m be any function verifying $m_1(x) \leq m(x) \leq m_2(x)$, for all $x \in R_X$. For $j = 1, 2$, define the random variables

$$\varepsilon_{j0} = Y_j - m(X_j),$$

which can also be expressed as

$$\varepsilon_{j0} = \varepsilon_j + (m_j(X_j) - m(X_j)).$$

Obviously, under the null hypothesis, $m_1(x) = m(x) = m_2(x)$, and $\varepsilon_{j0} = \varepsilon_j$. However under the alternative hypothesis, it happens that

$$E(\varepsilon_{10}) < 0 \qquad \text{and} \qquad E(\varepsilon_{20}) > 0.$$

Therefore, the comparison of the expectations of the regression errors under the null hypothesis can be used to detect the alternative hypothesis H_1.

In practice, the regression errors need to be estimated based on samples. For $j = 1, 2$, let

$$\{(X_{ij}, Y_{ij}),\ i = 1, ..., n_j\},$$

be an i.i.d. sample from the distribution of (X_j, Y_j). Denote $n = n_1 + n_2$ for the total sample size. Let, for $j = 1, 2$,

$$\hat{m}_j(x) = \sum_{i=1}^{n_j} W_{ij}(x, h_n) Y_{ij} \tag{4}$$

be the estimator of the regression function, where

$$W_{ij}(x, h_n) = \frac{K((x - X_{ij})h_n^{-1})}{n h_n \hat{f}_{X_j}(x)}$$

are Nadaraya-Watson type weights, and

$$\hat{f}_{X_j}(x) = \frac{1}{n h_n} \sum_{i'=1}^{n_j} K((x - X_{i'j})h_n^{-1}) \tag{5}$$

denotes the kernel density estimator of the density, f_{X_j}, of X_j, K is a known kernel function (typically, a symmetric density), and h_n is an appropriate bandwidth sequence. Given a function p such that $0 \leq p(x) \leq 1$ for all $x \in R_X$, we consider

$$\hat{m}(x) = p(x)\hat{m}_1(x) + (1 - p(x))\hat{m}_2(x). \tag{6}$$

A short discussion about an optimal choice of the function p is included in Remark 4 below. Now, estimate the regression errors under the null hypothesis using the function $\hat{m}$, for $j = 1, 2$ and $i = 1, \ldots, n_j$,

$$\hat{\varepsilon}_{ij0} = Y_{ij} - \hat{m}(X_{ij}),$$

and consider the corresponding weighted averages, for $j = 1, 2$,

$$\bar{\varepsilon}_{j0} = \frac{1}{n_j} \sum_{i=1}^{n_j} \hat{\varepsilon}_{ij0} w_j(X_{ij}),$$

where w_j is a positive weight function. We propose the following test statistic

$$T = \left(\frac{n_1 n_2}{n} \right)^{1/2} (\bar{\varepsilon}_{20} - \bar{\varepsilon}_{10}),$$

which has asymptotic normal distribution, as stated in Theorem 1 (see Section 3). The null hypothesis is rejected for positive large values of the test statistic.

Note that $\hat{m}(x)$ is a consistent estimator for $m(x) = p(x)m_1(x) + (1 - p(x))m_2(x)$ and hence $\bar{\varepsilon}_{20} - \bar{\varepsilon}_{10}$ estimates

$$\begin{aligned} \Delta &= E[\varepsilon_{20} w_2(X_2) - \varepsilon_{10} w_1(X_1)] \\ &= \int (m_2(x) - m(x)) w_2(x) f_{X_2}(x)\, dx - \int (m_1(x) - m(x)) w_1(x) f_{X_1}(x)\, dx \\ &= \int (m_2(x) - m_1(x)) f(x)\, dx, \end{aligned} \tag{7}$$

where we have defined

$$f(x) = p(x) w_2(x) f_{X_2}(x) + (1 - p(x)) w_1(x) f_{X_1}(x). \tag{8}$$

Furthermore, note that T corresponds to the empirical process of residuals considered by Neumeyer and Dette (2003), evaluated in $t = 1$ with $R_X = [0, 1]$. These authors proposed Kolmogorov-Smirnov and Cramér-von Mises type statistics based on that empirical process to test for the equality of two regression curves versus a general alternative of non-equal curves. However, in the case of one-sided alternatives, the much simpler test statistic T with asymptotic normal law can be applied. Neumeyer and Dette (2003) used random weight functions $p(x) = n_1 \hat{f}_{X_1}(x)[n_1 \hat{f}_{X_1}(x) + n_2 \hat{f}_{X_2}(x)]^{-1}$ and $w_j(x) = (\hat{f}_{X_j}(x))^{-1}$,

$j = 1, 2$. The latter choice was motivated by a cancellation of bias terms that allowed these authors to use bandwidth rates optimal for regression estimation. This could be done in the definition of T as well. However, to keep the test statistic simple, we do not follow this approach, and give the asymptotic results under more restrictive bandwidth conditions. We further assume deterministic weight functions and discuss the applicability of random weight functions in Remark 4.

In the following result we obtain the asymptotic distribution of the test statistic under the alternative hypothesis H_1 and local alternatives of the type

$$H_{1n} : m_2(x) = m_1(x) + n^{-1/2} r(x), \text{ where } r(x) \geq 0 \text{ for all } x \in R_X, \tag{9}$$

which include the null hypothesis H_0 when $r \equiv 0$. Before stating the main result, we need to introduce the following regularity assumptions.

(A1) (i) The kernel function K is a symmetric density with compact support and twice continuously differentiable.

(ii) $nh_n \to \infty$ and $nh_n^4 \to 0$ as $n \to \infty$.

(A2) (i) For $j = 1, 2$, $n_j/n \to \kappa_j > 0$ as $n \to \infty$.

(ii) For $j = 1, 2$, the functions m_j, f_{X_j} and w_j, and p are twice continuously differentiable, σ_j is continuous and $\sigma_j(x) \geq C$ for some $C > 0$ and for all $x \in R_X$. R_X is a bounded interval.

(iii) For $j = 1, 2$, $E[\varepsilon_j^4] < \infty$.

Theorem 1 *Assume (A1) and (A2). Under H_{1n}, the asymptotic distribution of the test statistic T is $N((\kappa_1\kappa_2)^{1/2} d, \tau_0^2)$, where*

$$d = \int r(x) f(x)\, dx \qquad \text{and} \qquad \tau_0^2 = \int \Big(\frac{\kappa_2 \sigma_1^2(x)}{f_{X_1}(x)} + \frac{\kappa_1 \sigma_2^2(x)}{f_{X_2}(x)} \Big) f^2(x)\, dx,$$

and f is defined in (8).

Under H_1 the asymptotic distribution of $T - (n_1 n_2/n)^{1/2} \Delta$ is $N(0, \tau_1^2)$, where

$$\tau_1^2 = \int \Big[\frac{\kappa_2}{f_{X_1}(x)} (\sigma_1^2(x) + [m_1(x) - m_2(x)]^2) + \frac{\kappa_1}{f_{X_2}(x)} (\sigma_2^2(x) + [m_1(x) - m_2(x)]^2) \Big] f^2(x)\, dx - \Delta^2,$$

and Δ is defined in (7).

The proof of this theorem is deferred to the appendix.

Remark 1 The previous theorem gives the asymptotic distribution of the test statistic T under both null and alternative hypotheses. The null hypothesis correspond to $r(x) = 0$, which implies $d = 0$, and hence the asymptotic distribution of T is $N(0, \tau_0^2)$. Based on this asymptotic distribution under the null hypothesis, the test that rejects the null hypothesis H_0 against the alternative H_1 when the observed value of T is larger that $\tau_0 \, z_{1-\alpha}$ has asymptotic significance level α, where $z_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution.

On the other hand, note that $d \geq 0$, which means that the test can detect local alternatives converging to the null hypothesis at the parametric rate $n^{-1/2}$. This feature is also accomplished by other methods based on the estimation of the residuals. See Neumeyer and Dette (2003) and Pardo-Fernández *et al.* (2007).

Remark 2 The asymptotic variance of the test statistic, τ_0^2, depends on unknown functions. If we take into account the definition of $f(x)$ in equation (8), then τ_0^2 can be written as:

$$\tau_0^2 = \kappa_2 E\left[\frac{\sigma_1^2(X_2)}{f_{X_1}(X_2)} f(X_2)p(X_2)w_2(X_2)\right] + \kappa_1 E\left[\frac{\sigma_2^2(X_1)}{f_{X_2}(X_1)} f(X_1)(1-p(X_1))w_1(X_1)\right]$$

In practice, this quantity is estimated by taking the corresponding empirical averages, where κ_1 and κ_2 are replaced by n_1/n and n_2/n and the unknown functions σ_1^2, σ_2^2, f_{X_1} and f_{X_2} are replaced by their nonparametric estimators:

$$\begin{aligned} \hat{\tau}_0^2 \;\; &= \;\; \frac{1}{n}\sum_{i=1}^{n_2} \frac{\hat{\sigma}_1^2(X_{i2})}{\hat{f}_{X_1}(X_{i2})} \hat{f}(X_{i2})p(X_{i2})w_2(X_{i2}) \\ &\quad + \frac{1}{n}\sum_{i=1}^{n_1} \frac{\hat{\sigma}_2^2(X_{i1})}{\hat{f}_{X_2}(X_{i1})} \hat{f}(X_{i1})(1-p(X_{i1}))w_1(X_{i1}), \end{aligned} \tag{10}$$

where

$$\hat{\sigma}_j^2(x) = \sum_{i=1}^{n_j} W_{ij}(x, h_n) Y_{ij}^2 - \hat{m}_j^2(x), \tag{11}$$

$$\hat{f}(x) = p(x)w_2(x)\hat{f}_{X_2}(x) + (1-p(x))w_1(x)\hat{f}_{X_1}(x),$$

and $\hat{m}_j$ and $\hat{f}_{X_j}$ are given in (4) and (5), respectively.

The test that rejects the null hypothesis H_0 in favor of the alternative H_1 when the observed value of the test statistic T exceeds $\hat{\tau}_0\ z_{1-\alpha}$, where $\hat{\tau}_0 = +\sqrt{\hat{\tau}_0^2}$ and $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution, has asymptotic level α.

Remark 3 Under the regularity assumptions detailed above, the deterministic weight functions p, w_1 and w_2 can be replaced by kernel estimators without changing the asymptotic distribution of the test statistic (compare with the weight functions applied by Neumeyer and Dette, 2003). The same applies for weight functions $w_j(x) = 1/\hat{\sigma}_j(x)$, where $\hat{\sigma}_j(x)$ is obtained from (11), which produce a standardization of the residuals by the estimated standard deviation (see also Pardo-Fernández *et al.*, 2007). Under the assumptions (A1) and (A2) with twice continuously differentiable σ_j^2 $(j = 1, 2)$, the asymptotic distribution of T under the null hypothesis is $N(0, \tau_0^2)$, where now

$$\tau_0^2 = \int \Big(\frac{\kappa_2\sigma_1^2(x)}{f_{X_1}(x)} + \frac{\kappa_1\sigma_2^2(x)}{f_{X_2}(x)}\Big)\Big(\frac{p(x)f_{X_2}(x)}{\sigma_2(x)} + \frac{(1-p(x))f_{X_1}(x)}{\sigma_1(x)}\Big)^2\, dx.$$

Remark 4 We will now consider optimal choices of the weight functions p, w_1 and w_2 in terms of power against local or fixed alternatives. To this end we first consider H_{1n} in the special case $r \equiv 1$ and maximize the signal-to-noise ratio

$$\frac{d^2}{\tau_0^2} \quad = \quad \frac{\left(\int f\right)^2}{\int gf^2}, \tag{12}$$

where the function g is defined as $\kappa_2\sigma_1^2/f_{X_1} + \kappa_1\sigma_2^2/f_{X_2}$ for the sake of simple notation. In the following we use the subindex $*$ for functions and values applying special weight functions p_*, w_{1*} and w_{2*}. The ratio (12) is only influenced by the weights through the function f defined in (8) and is maximized by the choice $f_* = 1/g$. This is shown by an application of Cauchy-Schwarz's inequality:

$$\frac{d^2}{\tau_0^2} \quad = \quad \frac{\left(\int \sqrt{g}f\cdot\frac{1}{\sqrt{g}}\right)^2}{\int gf^2} \quad \le \quad \frac{\int gf^2 \int \frac{1}{g}}{\int gf^2} \quad = \quad \int f_* \quad = \quad \frac{\left(\int f_*\right)^2}{\int gf_*^2} \quad = \quad \frac{d_*^2}{\tau_{0*}^2}.$$

This optimal ratio can be achieved by the following weight functions (where we replace unknown functions by their estimators according to Remark 2):

$$p_*(x) = \frac{n_1\hat{f}_{X_1}(x)\hat{\sigma}_1^{-2}(x)}{n_1\hat{f}_{X_1}(x)\hat{\sigma}_1^{-2}(x) + n_2\hat{f}_{X_2}(x)\hat{\sigma}_2^{-2}(x)} \tag{13}$$

$$w_{1*}(x) = \frac{1}{\hat{\sigma}_1^2(x)}, \qquad w_{2*}(x) \;=\; \frac{1}{\hat{\sigma}_2^2(x)}. \tag{14}$$

Please note that the estimator $\hat{m}_*(x) = p_*(x)\hat{m}_1(x) + (1 - p_*(x))\hat{m}_2(x)$ is the same regression estimator as was applied by Munk, Neumeyer and Scholz (2007) and, under H_0, it is the optimal regression estimator from the pooled sample with respect to the asymptotic mean squared error.

Now assume the fixed alternative H_1 is valid with $m_2 - m_1 \equiv 1$. We will obtain optimal weight functions with respect to power by maximizing the ratio

$$\frac{\Delta^2}{\tau_1^2} \;=\; \frac{\left(\int f\right)^2}{\int \tilde{g} f^2 - (\int f)^2},$$

where $\tilde{g} = g + \kappa_2/f_{X_1} + \kappa_1/f_{X_2}$. We show that the optimal function is $f_* = 1/\tilde{g}$. To this end note that $\Delta^2/\tau_1^2 \leq \Delta_*^2/\tau_{1*}^2$ is equivalent to

$$\left(\int f\right)^2 \int \tilde{g} f_*^2 \;\leq\; \left(\int f_*\right)^2 \int \tilde{g} f^2,$$

which is shown again by Cauchy-Schwarz's inequality:

$$\left(\int f\right)^2 \int \tilde{g} f_*^2 \;=\; \left(\int \sqrt{\tilde{g}} f \cdot \frac{1}{\sqrt{\tilde{g}}}\right)^2 \int f_* \;\leq\; \int \tilde{g} f^2 \int \frac{1}{\tilde{g}} \int f_* \;=\; \left(\int f_*\right)^2 \int \tilde{g} f^2.$$

Please note that we only have considered special cases of optimal weight functions, which in general would depend on the regression functions m_1 and m_2.

3 Simulations

In this section we present some simulation results to show the practical behaviour of the test proposed in Section 2. We consider the following models for the regression functions:

(i)	$m_1(x) = 1$	$m_2(x) = 1$
(ii)	$m_1(x) = x$	$m_2(x) = x$
(iii)	$m_1(x) = \sin(2\pi x)$	$m_2(x) = \sin(2\pi x)$
(iv)	$m_1(x) = x$	$m_2(x) = x + 0.25$
(v)	$m_1(x) = 1$	$m_2(x) = 1 + 0.5x$
(vi)	$m_1(x) = \sin(2\pi x)$	$m_2(x) = \sin(2\pi x) + 0.5x$

Models (i)–(iii) correspond to the null hypothesis, and models (iv)–(vi) correspond to the alternative hypothesis. We also consider homoscedastic and heteroscedastic models: in the case of homoscedasticity the variances are

$$\sigma_1^2(x) = 0.50^2 \quad \text{and} \quad \sigma_2^2(x) = 0.75^2, \tag{15}$$

whereas in the heteroscedastic case the variance functions are

$$\sigma_1^2(x) = (0.25 + 0.50x)^2 \quad \text{and} \quad \sigma_2^2(x) = (0.50 + 0.50x)^2. \tag{16}$$

We investigate the behaviour of the test under two types of distributions for the regression errors ε_1 and ε_2: standard normal and centered exponential. In all cases the covariates X_1 and X_2 have uniform distribution on $[0, 1]$. Tables in this section will display the rejection proportions in 1000 trials of the test, for simple sizes $(n_1, n_2) = (50, 50)$, $(50, 100)$ and $(100, 100)$, and significance levels $\alpha = 0.10$, 0.05 and 0.025.

The bandwidths required for the nonparametric estimation of regression, variance and density functions are chosen by a regular cross-validation procedure (see, for instance, Härdle, 1990). More precisely, in each population, we use cross-validation to obtain a bandwidth h_j to estimate m_j, and then the same bandwidth is used to estimate f_{X_j} and σ_j^2. For the kernel function needed in the nonparametric estimation we choose the kernel of Epanechnikov $K(u) = 0.75(1 - u^2)I(|u| < 1)$.

In Table 1 we study the impact of the choice of the function p, which is used in the estimation of the common regression function m. In principle, any function p satisfying $0 \leq p(x) \leq 1$ is valid. In Table we show the rejection proportions for models (ii) and (v) with the following simple choices: $p(x) = 0$, $p(x) = 0.25$, $p(x) = 0.50$, $p(x) = 0.75$ and $p(x) = 1$. In this case the regression errors have standard normal distribution. The results show a good approximation of the level in all cases. The behaviour of the power is also very similar for the different choices of p, thus it seems that the impact of the choice of p on the test is rather limited. In the rest of the tables we only show results for $p(x) = 0.5$.

Table 2 shows the rejection probabilities for models (i) to (vi), with homoscedastic variances. The distribution of the regression errors is $N(0, 1)$ in the top part of the table

Table 1: *Rejection probabilities under models (ii) and (v) for different choices of the function $p(x)$. The models are homoscedastic, with variances given in (15), and heteroscedastic, with variances given in (16). The distribution of the regression errors is $N(0,1)$.*

		(n_1, n_2) :	(50, 50)			(50, 100)			(100, 100)		
model	$p(x)$	α :	0.100	0.050	0.025	0.100	0.050	0.025	0.100	0.050	0.025
						Homoscedastic models					
(ii)	0		0.102	0.057	0.032	0.107	0.046	0.024	0.096	0.050	0.031
	0.25		0.105	0.058	0.032	0.108	0.046	0.027	0.102	0.053	0.031
	0.50		0.110	0.060	0.028	0.107	0.046	0.026	0.101	0.054	0.032
	0.75		0.113	0.061	0.031	0.107	0.049	0.027	0.102	0.055	0.031
	1		0.111	0.063	0.032	0.106	0.052	0.028	0.102	0.054	0.031
(v)	0		0.740	0.617	0.509	0.854	0.769	0.688	0.906	0.837	0.768
	0.25		0.742	0.626	0.512	0.857	0.772	0.687	0.909	0.840	0.776
	0.50		0.744	0.620	0.514	0.863	0.777	0.699	0.910	0.843	0.769
	0.75		0.738	0.615	0.510	0.862	0.772	0.694	0.911	0.844	0.773
	1		0.736	0.607	0.503	0.861	0.776	0.687	0.906	0.844	0.771
						Heteroscedastic models					
(ii)	0		0.107	0.057	0.029	0.102	0.053	0.032	0.099	0.055	0.029
	0.25		0.108	0.059	0.026	0.107	0.051	0.028	0.099	0.058	0.028
	0.50		0.111	0.060	0.025	0.109	0.050	0.028	0.098	0.060	0.030
	0.75		0.113	0.061	0.024	0.107	0.050	0.028	0.097	0.061	0.031
	1		0.110	0.060	0.026	0.109	0.049	0.029	0.096	0.061	0.032
(v)	0		0.732	0.610	0.491	0.838	0.753	0.659	0.902	0.825	0.748
	0.25		0.735	0.611	0.499	0.837	0.757	0.664	0.903	0.825	0.752
	0.50		0.737	0.614	0.493	0.842	0.759	0.667	0.904	0.830	0.756
	0.75		0.736	0.608	0.493	0.846	0.753	0.667	0.901	0.829	0.762
	1		0.732	0.601	0.486	0.848	0.748	0.666	0.898	0.828	0.764

and $Exponential(1) - 1$ in the bottom part of the table. The approximation of the level –models (i) to (iii)– is good in most cases. As expected, the power –models (iv) to (vi)– increases as the sample sizes increase. Table 3 shows the corresponding results for heteroscedastic models. Similar conclusions can be stated in this case.

For the sake of comparison with other existing procedures to test for one-sided alternatives, we considered the test proposed by Neumeyer and Dette (2005). This test is a signed-ranks-test of the residuals. The test statistic is

$$U = \sqrt{n} \left\{ \frac{1}{n_2 n} \sum_{i=1}^{n_2} \left(\sum_{k=1}^{n_1} I(\tilde{\varepsilon}_{k1} \leq \tilde{\varepsilon}_{i2}) + \sum_{k=1}^{n_2} I(\tilde{\varepsilon}_{k2} \leq \tilde{\varepsilon}_{i2}) \right) - 0.5 \right\},$$

where, for $j = 1, 2$ and $i = 1, \dots, n_j$,

$$\tilde{\varepsilon}_{ij} = \frac{Y_{ij} - \tilde{m}(X_{ij})}{\hat{\sigma}_j(X_{ij})},$$

Table 2: *Rejection probabilities under models (i)–(vi), with* $p(x) = 0.5$*. The models are homoscedastic with variances given in (15). The distribution of the regression errors is* $N(0,1)$ *(top) and* $Exponential(1) - 1$ *(bottom).*

	(n_1, n_2) :	(50, 50)			(50, 100)			(100, 100)		
model	α :	0.100	0.050	0.025	0.100	0.050	0.025	0.100	0.050	0.025
					$\varepsilon_1, \varepsilon_2 \sim N(0,1)$					
(i)		0.111	0.061	0.034	0.110	0.049	0.028	0.102	0.052	0.033
(ii)		0.110	0.060	0.028	0.107	0.046	0.026	0.101	0.054	0.032
(iii)		0.093	0.046	0.023	0.100	0.044	0.019	0.082	0.041	0.024
(iv)		0.735	0.610	0.489	0.858	0.769	0.674	0.908	0.836	0.759
(v)		0.744	0.620	0.514	0.863	0.777	0.699	0.910	0.843	0.769
(vi)		0.698	0.560	0.426	0.821	0.718	0.595	0.887	0.808	0.713
					$\varepsilon_1, \varepsilon_2 \sim Exponential(1) - 1$					
(i)		0.092	0.045	0.018	0.097	0.059	0.022	0.095	0.046	0.024
(ii)		0.090	0.042	0.018	0.096	0.054	0.019	0.091	0.044	0.021
(iii)		0.090	0.037	0.017	0.080	0.038	0.020	0.086	0.034	0.012
(iv)		0.769	0.619	0.497	0.875	0.777	0.675	0.937	0.883	0.811
(v)		0.773	0.651	0.528	0.873	0.785	0.692	0.938	0.884	0.827
(vi)		0.708	0.559	0.428	0.832	0.716	0.605	0.915	0.840	0.743

Table 3: *Rejection probabilities under models (i)–(vi), with* $p(x) = 0.5$*. The models are heteroscedastic with variances given in (16). The distribution of the regression errors is* $N(0,1)$ *(top) and* $Exponential(1) - 1$ *(bottom).*

	(n_1, n_2) :	(50, 50)			(50, 100)			(100, 100)		
model	α :	0.100	0.050	0.025	0.100	0.050	0.025	0.100	0.050	0.025
					$\varepsilon_1, \varepsilon_2 \sim N(0,1)$					
(i)		0.110	0.061	0.028	0.108	0.052	0.030	0.108	0.061	0.032
(ii)		0.111	0.060	0.025	0.109	0.050	0.028	0.098	0.060	0.030
(iii)		0.099	0.046	0.017	0.100	0.045	0.024	0.087	0.043	0.021
(iv)		0.732	0.593	0.477	0.842	0.745	0.643	0.895	0.826	0.740
(v)		0.737	0.614	0.493	0.842	0.759	0.667	0.904	0.830	0.756
(vi)		0.678	0.541	0.408	0.808	0.704	0.567	0.878	0.793	0.691
					$\varepsilon_1, \varepsilon_2 \sim Exponential(1) - 1$					
(i)		0.094	0.048	0.020	0.119	0.057	0.033	0.095	0.048	0.023
(ii)		0.090	0.045	0.020	0.103	0.053	0.030	0.092	0.045	0.023
(iii)		0.083	0.032	0.018	0.087	0.041	0.022	0.089	0.035	0.015
(iv)		0.747	0.615	0.493	0.855	0.770	0.658	0.930	0.863	0.782
(v)		0.754	0.640	0.504	0.865	0.768	0.671	0.931	0.873	0.801
(vi)		0.701	0.552	0.409	0.813	0.689	0.576	0.909	0.822	0.715

Table 4: *Rejection probabilities under models (i)–(vi) for the test proposed by Neumeyer and Dette (2005). The models are heteroscedastic, with variances given in (16). The distribution of the regression errors is $N(0,1)$ (top) and $Exponential(1) - 1$ (bottom).*

	(n_1, n_2) :	(50, 50)			(50, 100)			(100, 100)		
model	α :	0.100	0.050	0.025	0.100	0.050	0.025	0.100	0.050	0.025
					$\varepsilon_1, \varepsilon_2 \sim N(0,1)$					
(*i*)		0.125	0.085	0.043	0.126	0.078	0.042	0.121	0.070	0.038
(*ii*)		0.130	0.077	0.051	0.139	0.070	0.037	0.123	0.076	0.041
(*iii*)		0.110	0.072	0.042	0.134	0.072	0.040	0.125	0.078	0.045
(*iv*)		0.773	0.672	0.560	0.900	0.842	0.764	0.934	0.887	0.834
(*v*)		0.642	0.532	0.425	0.801	0.704	0.603	0.840	0.757	0.668
(*vi*)		0.650	0.540	0.412	0.803	0.703	0.590	0.860	0.793	0.693
					$\varepsilon_1, \varepsilon_2 \sim Exponential(1) - 1$					
(*i*)		0.321	0.227	0.162	0.308	0.239	0.164	0.432	0.362	0.291
(*ii*)		0.323	0.229	0.161	0.319	0.241	0.169	0.434	0.362	0.282
(*iii*)		0.174	0.110	0.072	0.211	0.148	0.093	0.284	0.210	0.145
(*iv*)		0.884	0.843	0.786	0.950	0.918	0.882	0.984	0.976	0.966
(*v*)		0.830	0.773	0.698	0.912	0.874	0.819	0.959	0.942	0.914
(*vi*)		0.743	0.650	0.561	0.875	0.810	0.736	0.946	0.911	0.866

$$\tilde{m}(x) = \frac{\sum_{j=1}^{2} \sum_{i=1}^{n_j} K((x - X_{ij})h^{-1}) Y_{ij}}{\sum_{j=1}^{2} \sum_{i=1}^{n_j} K((x - X_{ij})h^{-1})},$$

and $\hat{\sigma}_j$ is given in (11). The distribution of this statistic is approximated by means of a symmetric wild bootstrap procedure.

Table 4 displays the rejection probabilities when the statistic U is used. We keep the setups proposed in Neumeyer and Dette (2005): the bandwidths are based on the estimator of the integrated variance function, and the critical values are obtained from 100 bootstrap replications. In the case of normal errors, the test seems to slightly overestimate the level. The power behaves correctly. If we compare with Table 3, we see that the test proposed in Section 2 has better power than the test based on U in models (v) and (vi), but it has worse power in model (iv). In the bottom part of Table 4, we can see that the level is not at all respected when the distribution of the errors is exponential. As explained in Neumeyer and Dette (2005), this test is only valid when the error distribution is symmetric. Nevertheless, this drawback could be avoided if a smooth bootstrap of residuals is used, as shown in Neumeyer (2008).

In Remark 4 we discussed the possibility of choosing the functions p, w_1 and w_2 in an optimal way in order to maximize the power of the test against alternatives of the form

Table 5: *Rejection probabilities under models* (i)*–*(iii) *and* (vii)*–*(ix)*. The models are heteroscedastic, with variances given in (16). The distribution of the regression errors is* $N(0,1)$ *(top) and* $Exponential(1)-1$ *(bottom).*

model	(n_1, n_2) :	(50, 50)			(50, 100)			(100, 100)		
	α :	0.100	0.050	0.025	0.100	0.050	0.025	0.100	0.050	0.025
					$\varepsilon_1, \varepsilon_2 \sim N(0,1)$					
(i)	Optimal	0.106	0.064	0.039	0.104	0.047	0.026	0.111	0.053	0.030
(ii)	Optimal	0.144	0.090	0.049	0.144	0.072	0.046	0.152	0.078	0.048
(iii)	Optimal	0.107	0.059	0.031	0.100	0.055	0.026	0.114	0.052	0.029
(vii)	Optimal	0.778	0.654	0.536	0.911	0.830	0.733	0.937	0.875	0.810
(vii)	Naive	0.740	0.601	0.498	0.854	0.758	0.657	0.900	0.834	0.754
$(viii)$	Optimal	0.817	0.709	0.598	0.928	0.863	0.779	0.959	0.921	0.854
$(viii)$	Naive	0.732	0.593	0.477	0.842	0.745	0.643	0.895	0.826	0.740
(ix)	Optimal	0.730	0.604	0.480	0.857	0.770	0.644	0.905	0.830	0.743
(ix)	Naive	0.683	0.539	0.406	0.800	0.701	0.568	0.879	0.790	0.691
					$\varepsilon_1, \varepsilon_2 \sim Exponential(1)-1$					
(i)	Optimal	0.089	0.040	0.022	0.113	0.062	0.032	0.088	0.036	0.020
(ii)	Optimal	0.143	0.086	0.049	0.184	0.112	0.074	0.157	0.092	0.049
(iii)	Optimal	0.085	0.043	0.019	0.101	0.049	0.025	0.090	0.042	0.015
(vii)	Optimal	0.786	0.664	0.542	0.918	0.851	0.781	0.958	0.903	0.850
(vii)	Naive	0.756	0.632	0.517	0.862	0.776	0.681	0.934	0.872	0.797
$(viii)$	Optimal	0.813	0.716	0.602	0.923	0.873	0.801	0.962	0.929	0.881
$(viii)$	Naive	0.747	0.615	0.493	0.855	0.770	0.658	0.930	0.863	0.782
(ix)	Optimal	0.697	0.557	0.414	0.876	0.790	0.685	0.915	0.856	0.742
(ix)	Naive	0.695	0.551	0.404	0.813	0.699	0.574	0.911	0.820	0.706

$m_2(x) = m_1(x) + c$, where c is a positive constant. To end our simulation study, we will show the results obtained with the choices proposed in (13) and (14) for p and w_1, w_2. We reconsider model (i)–(iii) to check the level approximation, and introduce the following scenarios to simulate the power behaviour:

$$\begin{array}{lll} (vii) & m_1(x) = 1 & m_2(x) = 1.25 \\ (viii) & m_1(x) = x & m_2(x) = x + 0.25 \\ (ix) & m_1(x) = \sin(2\pi x) & m_2(x) = \sin(2\pi x) + 0.25 \end{array}$$

Table 5 displays the simulated rejection probabilities based on heteroscedastic models, with variances given in (16). Rows with ‘Optimal’ contain the results with the choices for p, w_1 and w_2 proposed in (13) and (14), which we call “optimal choices”; rows with ‘Naive’ contain the results with the “naive choices” $p(x) = 0.5$ and $w_1(x) = w_2(x) = 1$. The level is well approximated with the “optimal choices” in models (i) and (iii), but it is overestimated in model (ii) (compare with Table 3). In power, the “optimal choices” produce a general improvement of about 10% with respect to the "naive choices", which is a clear advantage. On the other hand, we should also point out two drawbacks of the "optimal choices": first, they involve the estimation of more functions to perform the test, and, second, they are "optimal" only for detecting parallel deviations from the null hypothesis.

4 An illustration to a data set

We illustrate our testing procedure by means of an application to a data set from the Data Archive of the *Journal of Applied Econometrics*. The observations consist of annual expenditures (from October 1986 to September 1987) of Dutch households. The data are registered in Dutch guilders (former currency of the Netherlands, before the introduction of the Euro). We will study the relation between the total annual expenditure and the expenditure on food of a household. Einmahl and Van Keilegom (2008) verified that model (1) holds when X='log of the total expenditure' is considered as a covariate and Y='log of the expenditure on food' is the response variable (even a homoscedastic model is verified).

We consider the following populations: households consisting of two persons (159 observations), households consisting of three persons (45 observations) and households consisting of four persons (73 observations). Figure 1 shows the scatter plots and estimated regression curves based on the cross-validation bandwidths.

Pardo-Fernández *et al.* (2007) tested for the equality of the three regression curves and by groups of two regression curves versus general alternatives. However, in this example, we think that it makes perfect sense to focus on one-sided alternatives. Take, for instance, the populations consisting of households of 2 and 3 persons. We would expect that the average of the food expenditure in the first population (households of 2 persons) is less (or at least not greater) than the food expenditure of the second population (households of 3 persons). In other words, intuitively, the food expenditure is expected to increase as the

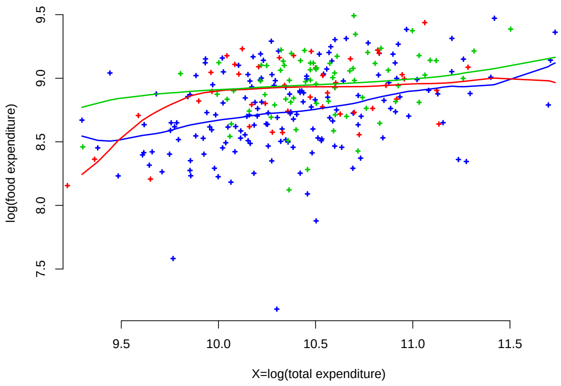

Figure 1: Scatter plot of Y='log(expenditure on food)' versus X='log(total expenditure)' of households of 2 members (in blue, 159 observations), 3 members (in red, 45 observations) and 4 members (in green, 73 observations).

Table 6: Data analysis (observed test statistics and p-values).

	observed test statistic	p-value
Population 1: households of 2 persons Population 2: households of 3 persons	$t = 0.761$	0.002
Population 1: households of 2 persons Population 2: households of 4 persons	$t = 1.360$	0.000
Population 1: households of 3 persons Population 2: households of 4 persons	$t = 0.323$	0.092

number of persons in the household increase, and this information should be incorporated to the test by means of one-sided alternatives.

We have then performed the test proposed in Section 2. The results are summarized in Table 6. The observed p-values show that the equality of the regression curves is clearly rejected in the cases of households of 2 and 3 persons, and 2 and 4 persons. When comparing households of 3 and 4 persons, the p-value is 0.092, so the equality of the corresponding regression curves could be accepted. This results are in agreement with the findings in Pardo-Fernández *et al.* (2007).

Appendix: Proof of Theorem 1

Proof. To ease notation define $p_1(x) = p(x)$, $p_2(x) = 1 - p(x)$ and let $j' = 3 - j$ for $j \in \{1, 2\}$. Then we have the expansion

$$\begin{aligned}
\bar{\varepsilon}_{j0} &= \frac{1}{n_j}\sum_{i=1}^{n_j}(Y_{ij} - \hat{m}(X_{ij}))w_j(X_{ij}) \\
&= \frac{1}{n_j}\sum_{i=1}^{n_j}\Big(\sigma_j(X_{ij})\varepsilon_{ij} + \sum_{l=1}^{2} p_l(X_{ij})[m_j(X_{ij}) - \hat{m}_l(X_{ij})]\Big)w_j(X_{ij}) \\
&= \frac{1}{n_j}\sum_{i=1}^{n_j}\sigma_j(X_{ij})\varepsilon_{ij}w_j(X_{ij}) \\
&\quad + \sum_{l=1}^{2}\frac{1}{n_l}\sum_{k=1}^{n_l}\frac{1}{n_j}\sum_{i=1}^{n_j}\frac{\frac{1}{h_n}K(\frac{X_{ij}-X_{kl}}{h_n})}{\hat{f}_{X_l}(X_{ij})}p_l(X_{ij})[m_j(X_{ij}) - Y_{kl}]w_j(X_{ij})
\end{aligned}$$

Under the assumptions of the theorem the kernel density estimators in the denominators can be replaced by the true densities and some means can be replaced by their expectations with negligible remainder terms. We obtain $\bar{\varepsilon}_{j0} = E_{j,n} + \Delta_{j,n} + o_p(n_j^{-1/2})$, where

$$\begin{aligned}
E_{j,n} &= \frac{1}{n_j}\sum_{i=1}^{n_j}\sigma_j(X_{ij})\varepsilon_{ij}w_j(X_{ij}) \\
&\quad - \sum_{l=1}^{2}\frac{1}{n_l}\sum_{k=1}^{n_l}\sigma_l(X_{kl})\varepsilon_{kl}\frac{1}{n_j}\sum_{i=1}^{n_j}\frac{\frac{1}{h_n}K(\frac{X_{ij}-X_{kl}}{h_n})}{f_{X_l}(X_{ij})}p_l(X_{ij})w_j(X_{ij}) + o_p(n_j^{-1/2}) \\
&= \frac{1}{n_j}\sum_{i=1}^{n_j}\sigma_j(X_{ij})\varepsilon_{ij}w_j(X_{ij}) \\
&\quad - \sum_{l=1}^{2}\frac{1}{n_l}\sum_{k=1}^{n_l}\sigma_l(X_{kl})\varepsilon_{kl}\int\frac{1}{h_n}K\Big(\frac{x - X_{kl}}{h_n}\Big)p_l(x)w_j(x)\frac{f_{X_j}(x)}{f_{X_l}(x)}\,dx + o_p(n_j^{-1/2}) \\
&= \frac{1}{n_j}\sum_{i=1}^{n_j}\sigma_j(X_{ij})\varepsilon_{ij}w_j(X_{ij}) - \sum_{l=1}^{2}\frac{1}{n_l}\sum_{k=1}^{n_l}\sigma_l(X_{kl})\varepsilon_{kl}p_l(X_{kl})w_j(X_{kl})\frac{f_{X_j}(X_{kl})}{f_{X_1}(X_{kl})} \\
&\quad + o_p(n_j^{-1/2})
\end{aligned}$$

and

$$\Delta_{j,n} = \frac{1}{n_{j'}}\sum_{k=1}^{n_{j'}}\frac{1}{n_j}\sum_{i=1}^{n_j}\frac{\frac{1}{h_n}K(\frac{X_{ij}-X_{kj'}}{h_n})}{f_{X_{j'}}(X_{ij})}p_{j'}(X_{ij})w_j(X_{ij})[m_j(X_{ij}) - m_{j'}(X_{kj'})].$$

Under the local alternatives H_{1n}, one obtains with standard argumentations for kernel estimation

$$\Delta_{j,n} \;=\; n^{-1/2}\int f_{X_j}(x)p_{j'}(x)w_j(x)(-1)^j r(x)\,dx + o_p(n^{-1/2}). \tag{17}$$

This gives for the test statistic under H_{1n}

$$T \;=\; \Big(\frac{n_1n_2}{n}\Big)^{1/2}(E_{2,n}-E_{1,n}) + (\kappa_1\kappa_2)^{1/2}d + o_p(1),$$

where

$$\begin{aligned}
E_{2,n}-E_{1,n} \;&=\; \frac{1}{n_2}\sum_{k=1}^{n_2}\sigma_2(X_{k2})\varepsilon_{k2}\Big[p_1(X_{k2})w_2(X_{k2}) + p_2(X_{k2})w_1(X_{k2})\frac{f_{X_1}(X_{k2})}{f_{X_2}(X_{k2})}\Big]\\
&\quad - \frac{1}{n_1}\sum_{k=1}^{n_1}\sigma_1(X_{k1})\varepsilon_{k1}\Big[p_1(X_{k1})w_2(X_{k1})\frac{f_{X_2}(X_{k1})}{f_{X_1}(X_{k1})} + p_2(X_{k1})w_1(X_{k1})\Big]\\
&\quad + o_p(n^{-1/2})\\
&=\; \sum_{j=1}^{2}(-1)^j\frac{1}{n_j}\sum_{k=1}^{n_j}\sigma_j(X_{kj})\varepsilon_{kj}\frac{f(X_{kj})}{f_{X_j}(X_{kj})} + o_p(n^{-1/2})
\end{aligned} \tag{18}$$

and the assertion under H_{1n} follows from the Central Limit Theorem for triangular arrays applying Lyapunov's condition.

Under H_1 we further consider

$$T - \Big(\frac{n_1n_2}{n}\Big)^{1/2}\Delta = \Big(\frac{n_1n_2}{n}\Big)^{1/2}(E_{2,n}-E_{1,n}) + \Big(\frac{n_1n_2}{n}\Big)^{1/2}(\Delta_{2,n}-\Delta_{1,n}-\Delta) + o_p(1), \tag{19}$$

where for $E_{2,n}-E_{1,n}$ the expansion (18) is valid also under H_1 and from (17) it follows that

$$\begin{aligned}
\Delta_{2,n}-\Delta_{1,n}-\Delta \;&=\; \frac{1}{n_1n_2}\sum_{k=1}^{n_1}\sum_{i=1}^{n_2}\frac{1}{h_n}K\Big(\frac{X_{i2}-X_{k1}}{h_n}\Big)[m_2(X_{i2})-m_1(X_{k1})]\\
&\quad \times\Big(\frac{p_1(X_{i2})w_2(X_{i2})}{f_{X_1}(X_{i2})} + \frac{p_2(X_{k1})w_1(X_{k1})}{f_{X_2}(X_{k1})}\Big) - \Delta
\end{aligned}$$

is a centered two-sample U-statistic with n-dependent kernel. Applying Hoeffding's decomposition and calculating the variances one can see that the degenerate U-statistic in the decomposition is negligible. Hence, one obtains

$$\begin{aligned}\Delta_{2,n}-\Delta_{1,n}-\Delta &= \frac{1}{n_1}\sum_{k=1}^{n_1}\int \frac{1}{h_n}K\Big(\frac{x-X_{k1}}{h_n}\Big)[m_2(x)-m_1(X_{k1})] \\ &\quad \times\Big(\frac{p_1(x)w_2(x)}{f_{X_1}(x)}+\frac{p_2(X_{k1})w_1(X_{k1})}{f_{X_2}(X_{k1})}\Big)f_{X_2}(x)\,dx \\ &\quad +\frac{1}{n_2}\sum_{i=1}^{n_2}\int \frac{1}{h_n}K\Big(\frac{X_{i2}-x}{h_n}\Big)[m_2(X_{i2})-m_1(x)] \\ &\quad \times\Big(\frac{p_1(X_{i2})w_2(X_{i2})}{f_{X_1}(X_{i2})}+\frac{p_2(x)w_1(x)}{f_{X_2}(x)}\Big)f_{X_1}(x)\,dx-\Delta+o_p(n^{-1/2}),\end{aligned}$$

which by standard arguments gives

$$\Delta_{2,n}-\Delta_{1,n}-\Delta = \sum_{j=1}^{2}\frac{1}{n_j}\sum_{k=1}^{n_j}[m_2(X_{kj})-m_1(X_{kj})]\frac{f(X_{kj})}{f_{X_j}(X_{kj})}-\Delta+o_p(n^{-1/2}). \quad (20)$$

Please note that here the dominating term is centered with respect to expectation. Now from (19) with (18) and (20) the assertion under H_1 follows applying the Central Limit Theorem. □

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