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in Logistic and Services Networks

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On the Weber Problem in Logistic and Services Networks

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Abstract: We investigate a queueing-location problem where a fixed number of customers cycle between a set of service stations and a central station, the location of which has to be determined in a way that the utilization of the given resources of the network is maximized (measured in throughput). We show that our objectives result in solving the well-known Weber problem. We discuss other performance objectives as well and the case that the demand at the service stations occur only in periods which are interrupted by no-demand periods.

Keywords: Gordon-Newell network, Weber problem, facility location, throughput, steady state analysis, travel times.

1 Introduction

We investigate the Weber problem in a dynamical environment which is determined by a logistic and services network. A typical location problem setting in our framework is as follows.

Introductory example: We have a set of stations (warehouses) where goods from a central station (production center) have to be delivered. The locations of the warehouses are fixed and known in advance.

Trucks are loaded at the central station, drive to one of the warehouses, are unloaded there, return to the central station, are loaded there again, drive to a warehouse, not necessarily the same as before, and so on. We take into consideration the distances between center and warehouses (measured as travel times), the loading and unloading times, and the delay due to queueing and waiting for loading and unloading, and the fraction of goods that have to be delivered to the warehouses.

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For a given number of trucks in the system we determine the location for the central production center such that the utilization of the given resources is maximized.

Maximizing the utilization of the given resources is characterized by maximization of the overall mean number of delivered goods per time unit, i.e., of the sum of the throughputs at the warehouses.

We also discuss other optimization criteria like transportation times and transportation costs.

The methodological approach we use is an integration of queueing network theory and location theory.

The classical „Weber Problem“, sometimes called „Fermat-Weber Problem“, dates back as a basic geometrical problem to Battista Cavalieri, Pierre de Fermat and Evarista Torricelli. Application of the problem to facility location was introduced by Alfred Weber in [WE09], for a historical survey see [DKLSW04]. The Weber problem, which will be part of our framework, is as follows:

There are n locations in the plane whose coordinates $(a_{j1}, a_{j2}) \in \mathbb{R}^2, j = 1, \ldots, n$, are given and known in advance. To each of this locations (*demand points*) is associated a positive weight c_j, $j = 1, \ldots, n$. We are to find the “location of a further point” $(x_1, x_2) \in \mathbb{R}^2$ (*center point, median point*) which minimizes the sum of the weighted Euclidean distances $d_j(x_1, x_2) = ((x_1 - a_{j1})^2 + (x_2 - a_{j2})^2)^{1/2}$ between the demand points and the center point to place, that is

$$\sum_{j=1}^{n} c_j d_j(x_1, x_2) = \min! \quad \text{and determine} \quad x^* = \arg\min_{x \in \mathbb{R}^2} \sum_{j=1}^{n} c_j d_j(x_1, x_2)$$

The problem can be generalized to the multi-dimensional case and to generalized convex distance functions easily. Being a convex minimization problem it has at least one solution. A very efficient solution procedure was developed in [WE37] and is now called the Weiszfeld-algorithm. Although other algorithms have been developed since then, it is still a very prominent method for solving the problem algorithmically.

Algorithms have also been developed for different convex distance functions, and the Weiszfeld-procedure has been adapted to these frameworks.

Location problems under stochastic influences have been investigated since the 1970’s by many authors. The research on location problems in connection with queuing systems began in the mid of the 70’s with a publication by Larson, see [LA74], and was continued in particular by himself and Berman, see [BLC85] and [BLP87], for discrete location problems. For surveys, see the relevant chapters in the collections [DH04] and [MF90].

In [DSS90] and [SJD99] the authors considered several location problems in the plane for a mobile server, where clients’ (customers’) demand occurs in Poisson streams at different points in the plane. The service system of the mobile server is assumed to behave as an $M|M|1|\infty$, $M|G|1|\infty$ or $M|G|s|\infty$ queue.

The guiding principle of these problem solutions is to model the problem as a single station queueing system with possibly several service channels (single or multi server stations) and to incorporate “travel times“ of the “server“ to and from the clients into the service times for the requests of the clients. Queueing occurs at the server station only.

The contribution of the present paper is to investigate besides the central server the behavior of the clients (warehouses) in detail, and the behavior of the trucks when unloading at the warehouses, and when waiting for being unloaded there. The interaction of the central server and clients via the cycling customers (trucks) is dealt with in detail.

In abstract terms our contribution can be described as follows:
We investigate the location problem for the central server in a star-like generalized Gordon-Newell network where a fixed number of customers cycle between the central station and (varying with time) the (exterior) service stations. The problem is to find an optimal location for the central service station, where optimality is defined with respect to maximization of the exterior stations' throughput. (It will turn out that this is equivalent to maximize the overall throughput of the network.)

Guided by the introductory example we include into the optimization procedure generalized travel times or distances. The travel times of the trucks on the lane from the central station to the exterior stations and back are incorporated into the queueing network models as infinite server queues with service times which are either deterministic (no overtaking of trucks) or random with any distribution having the prescribed mean (overtaking of trucks is possible). Therefore, each branch of the network from the center to the exterior stations consists of three network nodes: one service station (= warehouse) and two infinite server stations (= roads to and from the station).
It turns out that we can incorporate into our framework rather general problem specifications of the introductory example by utilizing general network classes available in the literature and the insensitivity properties offered by queueing network theory.

For simplicity we will present in a first step the main development in terms of purely exponential networks, and will thereafter describe the transformation to more realistic models for the lanes and the loading and unloading stations. Exploiting the exponential assumptions we can utilize the simplest part of the theory of product form queueing networks, sketched in Section 3.1, to model and solve the location problem.

Our main finding is described and discussed in Section 2. For the queueing network and location model we are able to show that we can transform the throughput-maximization problem into a classical minisum Weber problem as convex optimization problem (we allow generalized convex distances).
It turns out, rather surprising to us because of being contra intuition, that the service times at the exterior nodes (the unloading times at the warehouses) are not relevant for locating the central station. Obviously, this is a strong insensitivity property of the problem, because we conclude that the (stochastic) properties of the unloading process can change over time without the optimal location becoming suboptimal.

Similarly, it turns out that the service capacity (loading capacity) at the center does not enter the optimization criterion for the location of the center, which is intuitive.

The only relevant information for the decision therefore are: The generalized distances between the center and the exterior nodes and the proportion of demand which has to be transported to the warehouses.

The consequences for managerial decisions are obvious:
The decision for the facility location can be decoupled from the fine-tuning of the stations' performance behavior.
Moreover, whenever we are in a position that we can enhance locally the behavior in the network (without reducing resources at other nodes) this will not decrease the network's throughput.
We discuss this and further consequences at the end of Section 2.

In Section 3.1 we provide the necessary prerequisits from product form queueing network theory, which enable us to formulate the introductory example (and similar problems) as star-like Gordon-Newell network in Section 3.2 and to compute the target throughput value for given location.

In Section 3.3 we discuss generalizations and extensions of the location problem, especially we show how to get rid of the modeling assumption of exponential distribution in Section 3.1. This is a rather easy consequence of the insensitivity theory for product form networks.

We consider further in Section 4.1 the case where the demand at the warehouses is varying over time. I.e., there are subsequent demand and no-demand periods of the warehouses. The structure of these sequences of alternating time intervals is very flexible for the modeling process. It turns out that a direct generalization of our previous results seems to be not possible. We therefore develop a heuristic approximation to model this framework which enables us to revert to product form network calculus. This yields again two results which are counter intuitive: The relevant variables for the location decision are only: The generalized distances between the center and the exterior nodes and the proportion of demand which has to be transported to the warehouses.
Additionally there is a strong insensitivity property of this decision procedure with respect to the local properties of the exterior nodes and their variation of demand over time.

While in Section 4.1 the demand and no-demand periods is a set-valued Markov process for its own, we discuss in Section 5 the case that the demand/no-demand behavior of the warehouses depends on the queue-length there, i.e., we allow that no further goods are requested for if the number of trucks waiting for unloading at the warehouse is above some threshold. It turns out, that besides some small extensions most of the nice problem structures from the previous sections are not maintained.

In Section 6 we summarize our findings and sketch directions of further research to continue the investigations of this paper.

Notation and conventions:
$\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$.
Empty sums are 0, empty products are 1, we set $0/0 = 0$.
For any set A we denote the set of all subsets of A by $\mathcal{P}(A)$.

2 Problem setting and main result

For the readers' convenience we describe the problem setting and our model in terms of the introductory example.

We have $J - 1$ (exterior) stations, the warehouses, numbered $j = 2, \dots, J$. Station j is located in the plane at coordinates $a_j = (a_{j1}, a_{j2}) \in \mathbb{R}^2$.

At station j there are $s_j \geq 1$ facilities (service channels) for unloading trucks, and there is ample waiting space for trucks that arrive while all unloading facilities are working. The duration of the unloading time is exponential with expectation μ_j^{-1}. The queueing regime for trucks is First-Come-First-Served (FCFS).

We are to find a central location $x = (x_1, x_2) \in \mathbb{R}^2$, where the central station, number 1, (production center) has to be placed. The distance between station j and the central station 1 is described by a convex function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+$. For simplicity we denote $d_j(x) := d(a_j, x)$.

At central station 1 there are $s_1 \geq 1$ facilities (service channels) for loading trucks, and there is ample waiting space for trucks that arrive while all loading facilities are working. The duration of the loading time is exponential with expectation μ_1^{-1}. The queueing regime for trucks is FCFS.

We abbreviate for station $j, j = 1, \dots, J$, if there are n_j trucks in service or waiting,

$$\mu_j(n_j) = \begin{cases} \mu_j \cdot n_j, & \text{if } \ n_j \leq s_j; \\ \mu_j \cdot s_j, & \text{if } \ n_j \geq s_j. \end{cases} \tag{2.1}$$

$\mu_j(n_j)$ can be interpreted as actual total service rate at station j, if there are n_j trucks in service or waiting.

A truck being loaded at the central station is directed to warehouse j with probability $\rho(j)$, $j = 2, \dots, J$, with $\sum_{j=2}^{J} \rho(j) = 1$. There are N identical trucks circulating in the system.

We assume that the service times are independent of one another and that the decision to which warehouse a truck is directed is independent of the history of the system.

Remark: We will show in the following sections that, given the location of the central, station the system can be described by standard closed queueing network models. There exists an ergodic Markov process which describes the time evolution of the system and has a unique steady state (equilibrium) distribution of the system.

We are interested in determining under equilibrium conditions the overall mean number of trucks leaving the warehouses per time unit. In terms of queueing networks this is exactly the sum of throughputs of all warehouse stations, which is a well defined performance measure.

Our aim is therefore to locate the central station 1 for delivering goods to the warehouses in such a way that the overall throughput of the warehouses is maximized.

Clearly, the throughput, as well as all network characteristics and performance measures will depend on the coordinates $x = (x_1, x_2)$ of the center. We henceforth indicate this dependence by writing in a system with N trucks and center location x for the throughput at the warehouses $TH_w(N; x)$. (The precise definition of throughput will be given below.)

Our optimization problem is to find

$$\max_{x=(x_1,x_2)\in\mathbb{R}^2}\{TH_w(N;x)\} \quad \text{and} \quad \arg\max_{x\in\mathbb{R}^2}\{TH_w(N;x)\}. \tag{2.2}$$

A little reflection yields the conjecture that the optimal location of the center will not depend on the loading capacity at the center, but will strongly depend on the generalized distances $d_j(x)$, the proportions (probabilities) $\rho(j), j = 2, \ldots, J$, for branching to the warehouses, and the unloading capacities at the warehouses.
In view of this our first theorem is to a certain extent counter-intuitive.

Theorem 2.1 *Let $x^* \in \mathbb{R}^2$ be a solution of the generalized Weber problem:*

$$\textit{Find} \quad \min_{x=(x_1,x_2)\in\mathbb{R}^2}\{\sum_{j=2}^{J}\rho(j)d_j(x)\} \quad \textit{and} \quad x^* = \arg\min_{x\in\mathbb{R}^2}\{\sum_{j=2}^{J}\rho(j)d_j(x)\} \tag{2.3}$$

Then x^* is a solution of the maximization problem (2.2) *as well.*

The proof is postponed to Appendix A.

The result of Theorem 1 is striking: With respect to the location of the center station for optimizing the overall warehouse throughput, the service capacities at the warehouses do not matter. The relevant information for the location decision is
• the (generalized) distances, which we express as travel times $d_j(x) := d(a_j, x)$, and
• the proportions $\rho(j)$ of goods to be dispatched to warehouse j.

Note: This does not mean that the throughput is independent of these local properties of the warehouses, as will be shown later there is a functional dependence.

The consequences of these observations for managerial decision making are obvious:
(1) We can at a first step make the decision for the location by way of a standard Weber problem, and thereafter perform fine-tuning according to the service resources at hand.
(2) Even more: If the system is established and built up, we can increase throughput by local enhancement of service.
(3) Furthermore: Shifting capacities between the nodes is possible without perturbing the optimality of the center's location as long as the fractions of demand for the stations

remain the same.
(4) Optimization of the number of trucks in the system for a target throughput can be made with the center location fixed already.

Saying it the other way round: Our Theorem 1 states that we can decouple (to a certain extent) the global decision for optimal location of the center station and local optimization of the processes performed at the warehouses and at the center station. E.g., it is intuitively clear that increasing the loading capacity at the center will increase the throughput at any warehouse (this is a consequence of Theorem 3.7 below).

It is less intuitive, that, if we maintain the capacities at the center and all but one warehouse and increase the unloading capacity at the remaining warehouse, the throughput at all warehouses increases. This is a consequence of Theorem 3.7 as well.

Remark: It may be of interest, to point out that our result encourages after fixing the location to enhance further the functioning of the local service. In the setting of our problem no analogue to Braess's paradox occurs, which proves that an "obvious" enhancement of transportation by opening of additional lanes may decrease throughput of the system.

Our conclusions are not subject to Braess's paradox because we have prescribed the lanes which are open for transportation and do not allow for short cutting.

We will discuss other optimization criteria in Section 3.4 below: Travel times and travel costs. Before doing this, we will transform the problem described so far into a queueing network model and discuss the modeling assumptions.

3 The star-like queueing network with central station

In terms of the introductory example the network model under consideration obviously carries a star-like structure with warehouses as exterior nodes and production center as center of the star. The network is closed with fixed number of trucks as customers.

The stations (warehouses as well as production centers) are nodes of the networks, the servers at the production center are the loading places, the servers at the warehouses are the unloading places, which all can operate in parallel, but there need not be ample capacity, so queueing may occur. Loading and unloading times are the service times of the servers and are random. The trucks are the customers circulating in the network.

For each warehouse, the roads from the center node to that warehouse and back are modeled as two additional nodes in the network with random travel times.

The variable decisions for the trucks' destinations on leave from the center is modeled as random decision where to go.

In terms of standard queueing network terms this model is a generalized Gordon-Newell network. For a shorter presentation and less complicated computations in the proofs we will develop our results in the setting of classical exponential Gordon-Newell networks. This requires some rather unrealistic assumptions, but it is easy to get rid of these assumptions and we will later on discuss why and how the results directly carry over to the general setting of the previous section.

For referencing we collect relevant results of network theory in the next section.

We assume that all random variables which occur in our models are defined on an underlying probability space $(\Omega, \mathcal{F}, P)$ which will not be mentioned further.

3.1 Prerequisits from stochastic network theory

Definition 3.1 (Gordon-Newell network) *Consider a set $\{1, 2, \ldots, J\}$ of service stations. Station j is a single server with state dependent service intensity and infinite waiting room under FCFS regime.*

There are $N > 0$ indistinguishable customers cycling according to an irreducible Markov matrix $R = (r(i,j), i, j = 1, \ldots, J)$. Customers arriving at node j request for an amount of work (service time) there which is exponentially distributed with mean 1. Whenever there are n_j customers present at node j (in service , if any, or waiting), service is provided with rate $\mu_j(n_j) > 0$. We assume that $\mu_j(n_j)$ is nondecreasing in n_j. All requested service times constitute an independent family of random variables.

Let $X_j(t)$ denote the number of customers present at station j at time $t \geq 0$, either waiting or in service (local queue length at station j). Then $X(t) := (X_j(t), j = 1, \ldots, J)$ is the joint queue length vector of the network at time t. We denote by $X = (X(t), t \geq 0)$ the joint queue length process of the Gordon-Newell network. States of X are

$$S(N, J) = \{(n_1, \ldots, n_J) \in \mathbb{N}^J, n_1 + \ldots + n_J = N\}.$$

Theorem 3.2 ([JA63], [GN67]) *The joint queue length process $X = (X(t) : t \in \mathbb{R}_+)$ of the Gordon-Newell network is a Markov process which is ergodic (irreducible and positive recurrent). Let $\eta = (\eta_1, \ldots, \eta_J)$ denote the unique probability solution of the traffic equation*

$$\eta_j = \sum_{i \in J} \eta_i r(i,j), \ j \in J. \tag{3.1}$$

The η_j are the customers' visit ratios at nodes $j = 1, \ldots, J$. The unique stationary and limiting distribution $\pi = \pi(N, J)$ of X on S(N,J) is

$$\pi(n_1, \ldots, n_J) = G^{-1}(N, J) \prod_{j=1}^{J} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)}, \ (n_1, \ldots, n_J) \in S(N, J),$$

where the norming constant is

$$G(N, J) := \sum_{n_1 + \ldots + n_J = N} \prod_{j=1}^{J} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)}.$$

Remark: The assumption that all service times are exponential-1 distributed is not a restriction. If at node j service times are exponential-ν_j then this is incorporated into the state dependent service rates by changing $\mu_j(n_j) \to \nu_j \cdot \mu_j(n_j)$.

Example 3.3 (Exponential multiservers) *If at node j for some $s_j \in \{1, 2, \ldots, \infty\}$ we have $\mu_j(n_j) = \mu_j \cdot \min(n_j, s_j)$, $n_j \geq 0$, $j \in \{1, 2, \ldots, J\}$ then the stochastic behaviour of the network's joint queue length process is identical to that of a Gordon-Newell network with the same number of nodes, routing structure, population size, but with multiserver nodes, where node j provides exponential-μ_j distributed service times to customers by at most s_j service channels.*
Especially, this example encompasses the case of infinite server queues, $s_j = \infty$. Such nodes are well suited to model traffic on lanes.

Lemma 3.4 [CY01] *For the Gordon-Newell network in equilibrium the arrival rate at any node equals the departure rate there. The mean number of departures per time unit from node j (node-j throughput) in the network with $N \geq 1$ customers is*

$$TH_j(N) := \sum_{(n_1,\ldots,n_J)\in S(N,J)} \pi(n_1, \ldots, n_J)\mu_j(n_j) = \eta_j \frac{G(N-1, J)}{G(N, J)}. \tag{3.2}$$

The (overall) throughput of the Gordon-Newell network is the sum of the local throughputs

$$TH(N) = \sum_{j=1}^{J} TH_j(N) = \frac{G(N-1, J)}{G(N, J)}. \tag{3.3}$$

Remark: It is possible to take as the solution of the traffic equation (3.1) any (not normalized) vector $\eta = (\eta_1, \ldots, \eta_J)$ which deviates from the normalized version by a factor. The stationary distribution π is invariant under changing the factor. If we take a not normalized η and plug it into the norming constants $G(N-1, J), G(N, J)$ then the throughput (3.3) reads

$$TH(N) = \sum_{j=1}^{J} TH_j(N) = \left(\sum_{j=1}^{J} \eta_j\right) \cdot \frac{G(N-1, J)}{G(N, J)}. \tag{3.4}$$

Theorem 3.5 (Little's Theorem, [CY01]) *Let in steady state denote $L_j(N)$ the mean number of customers present at node j and $W_j(N)$ the mean sojourn time of a customer at node j, then*

$$L_j(N) = \eta_j \cdot TH(N) \cdot W_j(N).$$

Due to $\sum_{j\in\bar{J}} L_j(N) = N$, *we get* $\sum_{j\in\bar{J}} \eta_j W_j(N) = \frac{N}{TH(N)}$.

Theorem 3.6 [WA89] *The throughput of a Gordon-Newell network according to Definition 3.1 is a nondecreasing monotone function in the number of customers: $TH(N) \geq TH(N-1)$, $N \in \mathbb{N}_+$.*

Theorem 3.7 **[SS94]** *Consider a Gordon-Newell network according to Definition 3.1. If at some node j the service rate is increased (point wise in the number of customers there) then the throughput at any node and in the network does not decrease.*

3.2 The model of the logistic and services network

We have $J-1$ (exterior) stations (the warehouses), each with one service channel (for unloading the trucks = customers), numbered $j = 2, \ldots, J$. Station j is placed in the plane at coordinates $a_j = (a_{j1}, a_{j2}) \in \mathbb{R}^2$ and a central station, number 1, at coordinates $x = (x_1, x_2) \in \mathbb{R}^2$, with one service channel for loading the trucks.

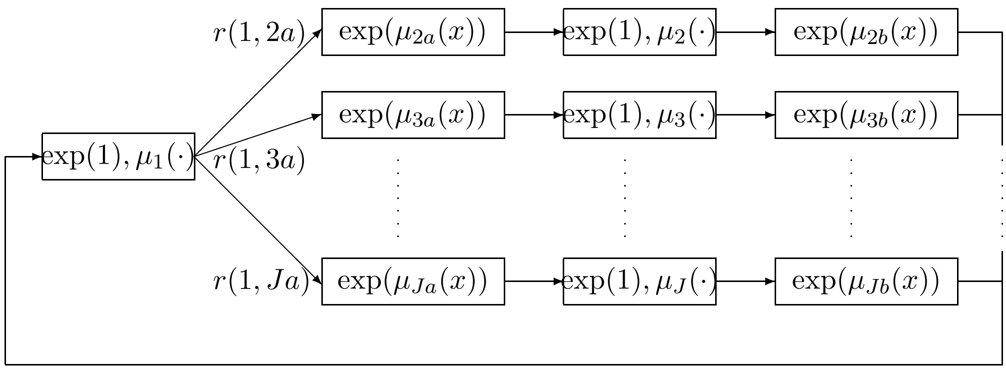
The distance between station j and the central station 1 is given by a convex function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+$. For simplicity we denote $d_j(x) := d(a_j, x)$.

We assume stations to have exponentially distributed service times with mean 1 and assume that service is provided with rate $\mu_j(n_j)$ if n_j customers (trucks) are present at station j. $N \in \mathbb{N}_+$ customers (trucks) are cycling in the network.

If a customer is served (loaded) at station 1, he will be routed to one of the other stations j and then has to travel distance $d_j(x)$.

Traveling this distance is represented as being served by an infinite server station with (for simplicity of presentation) exponentially distributed service times with mean $d_j(x)$. This infinite server is denoted ja. After passing the road (= being served by this server), the customer will be served at station j. When his service at warehouse station j is finished, he has to travel way back to station 1, which is represented again as being served by an infinite server station with exponentially distributed service times with mean $d_j(x)$. This infinite server station is denoted jb. The routing probability from server 1 to server ja is $r(1, ja) := \rho_j$, $j = 2, \ldots, J$, recall $\sum_{j=2}^{J} r(1, ja) = 1$, all other routing decisions are deterministic as indicated in the figure below.

With these stations and routing we have constructed a star-like network. We assume that all requested service times constitute an independent family of random variables. Routing decisions at node 1 are independent of the previous history of the network.



Thus, we have obtained a Gordon-Newell network with node set

$\bar{J} = \{1, 2a, 3a, \ldots, Ja, 2, 3, \ldots, J, 2b, 3b, \ldots, Jb\}$, state space

$$S(N, \bar{J}) = \{(n_j : j \in \bar{J}) \in \mathbb{N}^{\bar{J}} : \sum_{j \in \bar{J}} n_j = N\}, \tag{3.5}$$

service intensities $\exp(\mu_{ja}(x)) = \exp(\mu_{jb}(x)) = d_j(x)^{-1}$, and routing matrix

$$\mathcal{R}^{\bar{J}} = \begin{pmatrix} 0 & r(1,2a) & r(1,3a) & \cdots & r(1,Ja) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Remark: The assumption of exponentially distributed travel times can be removed, see the discussion in Section 3.3. Similarly, properties of the loading and unloading process and the distributional assumptions put on these times will be discussed as well. Obviously, this removal opens the way to construct much more realistic models. Fortunately enough, the formulas and the results derived here will maintain valid even in the more general setting. The reason behind is insensitivity theory for queueing networks.

Having fixed the model and the notation, we are now in a position to determine with the help of Section 3.1 the overall throughput at the warehouse stations.

Theorem 3.8 *Denote by $G(N, \bar{J}; x)$ the norming constant of the system if N trucks are cycling and the center station is located at $x \in \mathbb{R}^2$, then*

$$G(N, \bar{J}; x) = \sum_{n=0}^{N} \left[\left(\sum_{n_1 + \ldots + n_J = N - n} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \right) \frac{2^n}{n!} \left(\sum_{j=2}^{J} \eta_j d_j(x) \right)^n \right].$$

The total throughput of the network is

$$TH(N; x) = \frac{G(N-1, \bar{J}; x)}{G(N, \bar{J}; x)}. \tag{3.6}$$

The overall throughput at the warehouse stations is

$$TH_w(N; x) = TH(N; x) \sum_{j=2}^{J} \eta_j = \frac{1}{4} \cdot TH(N; x). \tag{3.7}$$

The proof is postponed to Appendix A.

3.3 Discussion of the modeling assumptions

Infinite server queues for modeling traffic on lanes are a standard device and offer a class of flexible models.

Assuming the travel times from center 1 to warehouse j and back via lanes ja and jb to be exponentially distributed with mean $d_j(x)$ enabled us to perform the proofs within the framework of exponential networks. This yields the most simple formulas for performance indices, e.g., throughput. We admit that exponential distribution is a quite unrealistic assumption for modeling traffic. Indeed, for given mean it is the extreme case with maximal entropy. Fortunately enough, we can remove this restriction and allow any distribution for the travel times with mean $d_j(x)$, obtaining the same throughput. E.g., taking deterministic-$d_j(x)$ times would model a constant flow of traffic without overtaking. This is the other extreme case with minimal entropy, not realistic either.

A realistic model for travel times of identical trucks is: Random mean-$d_j(x)$ travel times with a small variance, generating some overtaking. We obtain throughput (3.7).

Insensitivity theory for queueing networks is the reason behind this flexibility. This theory dates back to insensitivity in "Verallgemeinerte Bedienungsschemata" and the BCMP and Kelly networks. For a survey see [DA01][Section 9 and 10].

The most prominent observation in insensitivity theory is: "At a symmetric server the stationary queue length distribution is insensitive against variation of the service time distribution as long as the mean is fixed."

This implies especially that we can compute the stationary queue length distribution at such servers by taking exponential service time - the result is the same for other distributions with the same mean. Furthermore, this holds for the networks of Gordon-Newell type as well: At any symmetric node in the network we can vary the shape of the service time distribution, maintaining the mean, without the queue length distribution in steady state regime being perturbed.

The important observation for our investigation is that infinite servers are symmetric. And furthermore, it follows that the norming constants are the same with different service time distribution (with mean held constant) as with exponential distributed service times. With some computational effort it follows that the throughput obeys the same insensitivity property. As a consequence, the formulas (3.6) and (3.7) from Theorem 3.8 are the valid quantities for our decision analysis in the more general settings as well.

Single server stations under FCFS with state dependent rates for loading and unloading of the trucks are prescribed for ease of computations and may be substituted by different service schemes.
(i) If there is ample capacity and all trucks are served in parallel, we have infinite servers and can allow general service time distributions. The throughput will be the same under any shape of the service time distributions as long as the mean service time is hold fixed.
(ii) If we have at the center s_1 loading facilities, and loading a truck needs an exponential-μ_1 distributed time, we model this as an exponential multiserver queue with s_1 channels, similarly at exterior stations for unloading places. For details see Example 3.3.

(iii) If we have a prescribed, even queue length dependent, service capacity (service rate) at the central station, and if loading of all trucks present there is in parallel, the suitable service discipline is "Processor Sharing" (PS): If there are n_1 trucks at the center station each of them obtains a fraction of $1/n_1$ of the station's total capacity. The throughput of the system will not change if we have non-exponential distributions with the same mean - a consequence of insensitivity theory because PS is a symmetric service discipline.

Location of the center coincides with one of the warehouse locations: This is a possible scenario, because we have reduced the queueing-location problem to a purely location problem - at least as long as the decision making for the center's location has to be made. If the center's optimal position is $x = a_j$ then the distances (travel times) fulfill for this node $d_j(x) = 0$, and consequently, the service times at the infinite servers ja and jb are zero. This situation is covered by our framework, but the direct application of the model to our introductory example may be questionable. If necessary, to remedy this potential problem we may introduce different customer types (= trucks) for the service of different warehouses. We will not go into detail here, because this is part of ongoing investigations.

3.4 Minimizing passage-times and transportation costs

Reduction of throughput maximization to the standard minisum Weber problem by Theorem 2.1 opens a path to solving several further optimization problems in the logistic and services network via reduction as well. In this section we present some examples, the results hold for the generalized models discussed in Section 3.3 as well.

Our first example is to minimize the *expected passage-time* in the network, which is the mean travel time for a truck between two successive departures from the center station. The *expected passage-time* $Z_j(N)$ *to station* j is the sum of all expected sojourn times (waiting plus service times), which a customer (truck) spends waiting and in service at station j and traveling to and from j, i.e., his sojourns at the "stations" ja, j and jb, and thereafter at station 1. Let $W_i(N)$ be the expected sojourn time at station i with N customers in the system:

$$Z_j(N) = W_{ja}(N) + W_j(N) + W_{jb}(N) + W_1(N), \quad j = 2, \dots, J.$$

The sojourn times depend on the location x of the central server 1, we write $W_j(N; x)$ and $Z_j(N; x)$ instead of $W_j(N)$ and $Z_j(N)$, respectively.

The overall *expected passage-time* is

$$Z(N; x) := \sum_{j=2}^{n} r(1, ja) Z_j(N; x).$$

With $\eta_j = (1/4)r(1, ja)$, $j = 1, \ldots, J$ and $\eta_1 = \sum_{j=2}^{J} \eta_j$ we get

$$Z(N; x) = \sum_{j=2}^{J} r(1, ja) Z_j(N; x) = 4 \sum_{j=2}^{J} \eta_j Z_j(N; x) = 4 \sum_{j \in \bar{J}} \eta_j W_j(N; x).$$

From Little's Theorem (Theorem 3.5) we obtain

$$\sum_{j \in \bar{J}} \eta_j W_j(N; x) = \frac{N}{TH(N; x)}.$$

This yields the optimization problem: Find

$$\arg \min_{x \in \mathbb{R}^2} \left\{ Z(N; x) = 4 \frac{N}{TH(N; x)} \right\}.$$

So $Z(N; x)$ attains its minimum at points where $TH(N; x)$ is maximal and from Theorem 2.1 we conclude that $Z(N; x)$ attains its minimum at x^* (given in Theorem 2.1). Hence, the travel time problem is reduced to a standard Weber problem as well.

In our second example we compute transportation costs for delivering the goods. There are costs K_j per time unit associated with delivering goods to warehouse j , which occur for a truck on the way from a station 1 to station j and back, $j = 2, \ldots, J$. Obviously a similar approach as before via Little's Theorem is possible. But a little reflection shows that we can directly convert to a classical minisum Weber problem.

This is due to the fact that on the roads no queueing occurs: the lanes are modeled as infinite server queues, and therefore the travel times (=sojourn times at stations ja and jb) are just the service times there.

The mean service time at stations ja and jb is the (convex) general distance $d_j(x)$, so the mean costs occurring with transport from the center 1 to warehouse j is $2K_j d_j(x)$. So the problem is reduced to the Weber problem:

$$\text{Find} \qquad \arg \min_{x \in \mathbb{R}^2} \left\{ \sum_{j=2}^{J} \rho(j) K_j d_j(x) \right\}.$$

4 Networks with time varying demand

4.1 The model

In this section we consider the network from Section 2 with warehouses where demand is time varying and occurs only for times of random lengths (= demand periods), interrupted by times of random lengths (= no-demand periods). Under these side constraints we want to find a location for the center station for optimal utilizing the resources of the network. Our experience is that there is no smooth solution to the problem similar to

Theorem 2.1. We therefore develop an approximate model which we can solve explicitly by exploiting formulas from product form network calculus as in the previous sections.

The main assumption is:
(A1) Whenever there is no demand from station j the whole branch $[ja \to j \to jb]$ is frozen, i.e., traveling on the lanes is stopped and unloading at j, if there is any, is interrupted until a new demand period commences for j.

Our next assumption is natural:
(A2) Whenever stations $\{i_1, i_2, \ldots, i_n\} \subseteq \{2, 3, \ldots, J\}$ are in a no-demand period, a truck starting from the center 1 is directed to $j \in \{2, 3, \ldots, J\} \setminus \{i_1, i_2, \ldots, i_n\}$ with probability $r(1, ja)$ conditioned on this set of nodes being in a demand period.
Whenever it happens that all stations $\{2, 3, \ldots, J\}$ are in a no-demand period the center station is stalled as well. If after such period some stations enter their next demand period the center is immediately activated and the first truck can depart there after an exponential-μ_1 distributed residual loading time.
Remark: *Rerouting* [SD03][Section 6] according to probabilities $r(1, \cdot)$ conditioned on $\{2, 3, \ldots, J\} \setminus \{i_1, i_2, \ldots, i_n\}$ being in demand period is the *skipping regime.*

The alternating random sequences of demand and no-demand periods can be constructed in a very flexible manner. They are defined by intensities for starting demand and no-demand periods, which are of a rather general structure.
(A3) Take any pair of non-negative functions

$$A : \mathcal{P}(\{2, 3 \ldots, J\}) \to [0, \infty) \quad \text{and} \quad B : \mathcal{P}(\{2, 3 \ldots, J\}) \to [0, \infty),$$

subject to $A(\emptyset) = 1$ and $B(\emptyset) = 1$ and (recall $0/0 = 0$) for all $\bar{K} \subseteq \{2, 3 \ldots, J\}$

$$\frac{A(\bar{K} \cup \bar{G})}{A(\bar{K})} < \infty \quad \forall \bar{K} \cap \bar{G} = \emptyset \quad \text{and} \quad \frac{B(\bar{K})}{B(\bar{K} \setminus \bar{H})} < \infty \quad \forall \bar{H} \subseteq \bar{K}.$$

Let $\bar{K} \subseteq \{2, 3 \ldots, J\}$ be the set of the warehouse stations which are in a no-demand period. Then the stations in the set $\bar{I} \subseteq \{2, 3 \ldots, J\} \setminus \bar{K}$, $\bar{I} \neq \emptyset$ jointly commence a no-demand period with intensity

$$\alpha(\bar{K}, \bar{K} \cup \bar{I}) := \frac{A(\bar{K} \cup \bar{I})}{A(\bar{K})},$$

whereas for the stations in the set $\bar{H} \subseteq \bar{K}$ the no-demand periods jointly expire (and the next demand period commences) with intensity

$$\beta(\bar{K}, \bar{K} \setminus \bar{H})) := \frac{B(\bar{K})}{B(\bar{K} \setminus \bar{H})} \ ,$$

Our aim is again to maximize the overall throughput at the warehouses. To model and solve this problem we revert to the theory of stochastic networks with unreliable nodes

as developed in [SD03], for more details we refer to [SA06]. The connection to those models which provide integrated models for performance analysis and reliability assessment (shortly: performability theory) is easy to see:
Stations in a no-demand period are considered to be broken down in the reliability context, while stations in a demand period are up and can process service.

A Markov process description of the system's development uses state space

$$S(N,\bar{J}) \times \mathcal{P}(\{2,3\ldots,J\}). \tag{4.1}$$

The meaning of a generic state $(n_i, i \in \bar{J}, \bar{I}) \in S(N, \bar{J} \times \mathcal{P}(\{2,3\ldots,J\})$ is:
• the stations $i \in \bar{I}$ are in no-demand periods,
• at station $i \in \bar{J}$ there are n_i trucks, traveling or being loaded or unloaded (or waiting for this at active stations) if $i \in \{2,3\ldots,J\} \setminus \bar{I}$, or if $i \in \bar{I}$, they are frozen there.

It is easy to see that the state process of the network in this form is Markov. The stationary distribution $\pi = \pi(N,\bar{J})$ of this network process is [SD03]

$$\pi(n,\bar{I}) = \tilde{G}^{-1}(N,\bar{J})\ \frac{A(\bar{I})}{B(\bar{I})}\ \prod_{j\in\bar{J}}\prod_{i=1}^{n_j}\frac{\eta_j}{\mu_j(i)}\quad ,$$
$$n = (n_1,\ldots,n_J) \in S(N,\bar{J}),\ \ \bar{I} \subseteq \{2,3\ldots,J\}.$$

where $(\eta_j, j \in \bar{J})$ is the solution of the traffic equation (3.1) and $\tilde{G}^{-1}(N,\bar{J})$ the normalization constant.
From $\pi = \pi(N,\bar{J})$ we readily obtain the stationary probability $\pi(\bar{I}) = \pi(N,\bar{J})(\bar{I})$ that the subset $\bar{I} \subseteq \{2,3\ldots,J\}$ of the warehouses is in no-demand periods.
Because the stationary distribution depends of the location x of the central station, we write $\pi(n,\bar{I};x)$, $\tilde{G}^{-1}(N,\bar{J};x)$ and for the local throughputs $\widetilde{TH}_j(N;x)$ and $\widetilde{TH}_w(N;x)$ for the sum of the warehouse throughputs.

4.2 Throughput maximization

In Section 2 we reduced maximization of the network's throughput $TH(N;x)$ (and of the warehouse throughputs) to minimization of weighted generalized distances. We argued that result is somehow contra intuition because the unloading speeds at warehouses do not influence the optimal location of the center. On the other hand this leads us to conjecture that a similar reduction might be possible in the system with time varying demand, i.e., the optimal solution for the throughput maximization obeys an insensitivity property with respect to changing the characteristics of the demand and no-demand periods. Indeed we have found such an insensitivity property.

Theorem 4.1 *In the network described in Section 4.1 with center station located in* $x \in \mathbb{R}^2$ *the throughput of node (station)* $i \in \bar{J}$ *is*

$$\widetilde{TH}_i(N;x) = \eta_i \frac{G(N-1,J;x)}{G(N,J;x)} \left(\sum_{\bar{I} \subseteq \{2,3\dots,J\},\ i \notin \bar{I}} \frac{A(\bar{I})}{B(\bar{I})} \left(\sum_{\bar{K} \subset \{2,3\dots,J\}} \frac{A(\bar{K})}{B(\bar{K})} \right)^{-1} \right), \quad (4.2)$$

where $G(N-1,J;x), G(N,J;x)$ *are normalization constants in the network with time invariant demand, see Theorem 3.2, and the overall throughput of the network is*

$$\widetilde{TH}(N;x) = \sum_{i \in \bar{J}} \widetilde{TH}_i(N;x), \quad (4.3)$$

and the overall throughput of the warehouses is

$$\widetilde{TH}_w(N;x) = \frac{1}{4} \cdot \widetilde{TH}_i(N;x). \quad (4.4)$$

The optimal location $x^* \in \mathbb{R}^2$ *for the center station which maximizes the total throughput of the network, all local throughputs, and the sum of all throughputs at the warehouses as well, is the solution of the minisum Weber problem:*

$$\textit{Find} \qquad \min_{x \in \mathbb{R}^2} \{ \sum_{j=2}^{J} \rho(j) d_j(x) \}. \quad (4.5)$$

Proof: The local throughput (4.2) can be found in [SD03][Proposition 7.4] and (4.3), (4.4) are from additivity of rates, respectively direct from definition. This representation of the global throughput separates a factor which contains the distances from remainder terms which are not dependent on the distances. But the factor which contains the distances is the same as occurring in Theorem 2.1. □

Although the insensitivity has been proven, some comments may be in order:
(i) The result is still surprising. An intuitive guess would suggest that the optimal location moves towards stations which have, on average, shorter no-demand periods and away from those stations which have on average long no-demand periods. However, here is an argument from the hindsight: assume that station j has on average long no-demand periods, and therefore j - and also stations ja and jb - are frozen for long times. If the center station 1 would move away from j, the distance to j increases, which implies that the service rates of stations ja and jb decrease at the same time. According to Little's Theorem (Theorem 3.5), the average number of customers at station j and on the lanes ja and jb would increase. So, more trucks would be frozen for a longer time. This obviously decreases throughput.
(ii) Although the insensitivity of Theorem 4.1 is in line with the insensitivity in Theorem 2.1, there is a more critical point with the result here: As we pointed out at the beginning of this section our model is a heuristic modeling approximation for obtaining explicit throughput formulas which can be dealt with analytically. The argument in (i)

now shows that to a certain extent the insensitivity observed here may be due to this model specification, i.e., it may be partly an artefact.

An interesting question arises from considering the general framework of the performability models in [SD03] and [SA06]. For our present problem the models there offer to let the length of the demand periods and of the no-demand periods depend on the load of the stations, i.e., the number of trucks on the lanes and the queue lengths at the unloading station. It turns out that results similar to Theorems 2.1 and Theorem 4.1 are in general not provable, especially, insensitivity under changing the warehouse stations' characteristics no longer holds. For more details see the Section 5.

5 Throughput formulae for queue length dependent time varying demand

In this section we discuss the location problem when the length of the demand and no-demand periods depend on the load of the stations, i.e., the number of trucks on the lanes and the queue lengths at the unloading station. The general message is that results similar to Theorems 2.1 and Theorem 4.1 do not hold, especially, insensitivity under changing the warehouse characteristics does not hold.

The setting is now with more general alternating random sequences of demand and no-demand periods. They are defined by rather general intensities for starting demand and no-demand periods. The construction is as follows [SD03][Definition 3.1].

(A3') Take any pair of non-negative functions

$$A,\ B: \bigcup_{\bar{I}\subseteq\{2,3\dots,J\}} \left(\{\bar{I}\} \times \mathbb{N}^{|\bar{I}|}\right) \to [0,\infty)$$

subject to $A(\emptyset, n_i : i \in \emptyset) := B(\emptyset, n_i : i \in \emptyset) := 1$.

Let $\bar{K} \subseteq \{2,3\dots,J\}$ be the set of the warehouse stations which are in a no-demand period and let $(n_j : j \in \bar{J})$ be the joint queue lengths state.

Then the stations in the set $\bar{G} \subseteq \{2,3\dots,J\} \setminus \bar{K},\ \bar{G} \neq \emptyset$ jointly commence a no-demand period with intensity

$$\alpha(\bar{K}, \bar{K}\cup\bar{G}, n_i : i \in \bar{J}) := \frac{A(\bar{K}\cup\bar{G}, n_i : i \in \bigcup_{j\in\bar{K}\cup\bar{G}}\{ja, j, jb\})}{A(\bar{K}, n_i : i \in \bigcup_{j\in\bar{K}}\{ja, j, jb\})},$$

whereas for the stations in the set $\bar{H} \subseteq \bar{K}$ the no-demand periods jointly expire (and the next demand period commences) with intensity

$$\beta(\bar{K}, \bar{K}\setminus\bar{H}, n_i : i \in \bar{J}) := \frac{B(\bar{K}, n_i : i \in \bigcup_{j\in\bar{K}}\{ja, j, jb\})}{B(\bar{K}\setminus\bar{H}, n_i : i \in \bigcup_{j\in\bar{K}\setminus\bar{H}}\{ja, j, jb\})}.$$

We require all intensities $\alpha(\bar{K}, \bar{K} \cup \bar{G}, n_i : i \in \bar{J})$ and $\beta(\bar{K}, \bar{K} \setminus \bar{H}, n_i : i \in \bar{J})$ to be finite (recall $0/0 = 0$).

It is easy to see that an ergodic Markov process description for this system can be constructed with state space $S(N, \bar{J}) \times \mathcal{P}(\{2, 3 \ldots, J\})$, for more details see the comments to formula (4.1). The steady state distribution of this process is [SD03][Theorem 6.5]

$$\pi(n_1, \ldots, n_J, \bar{I}) = \frac{A(\bar{I}, n_i, i \in \bigcup_{j \in \bar{I}} \{ja, j, jb\})}{B(\bar{I}, n_i, i \in \bigcup_{j \in \bar{I}} \{ja, j, jb\})} \tilde{G}^{-1}(N, J) \prod_{j=1}^{J} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)},$$

with $(\eta_1, \ldots, \eta_J)$ the probability solution of the traffic equation (3.1) and the norming constant

$$\tilde{G}(N, J) := \sum_{(n_1, \ldots, n_J) \in S(N,J)} \left(\sum_{\bar{I} \subseteq \bar{J}} \frac{A(\bar{I}, n_i, i \in \bigcup_{j \in \bar{I}} \{ja, j, jb\})}{B(\bar{I}, n_i, i \in \bigcup_{j \in \bar{I}} \{ja, j, jb\})} \right) \prod_{j=1}^{J} \prod_{i=1}^{n_j} \frac{\eta_j}{\mu_j(i)}.$$

The node-j throughput in the network with $N \geq 1$ customers is

$$\widetilde{TH}_j(N) = \sum_{(n_1, \ldots, n_J) \in S(N,J)} \sum_{\bar{I}, \bar{J}, j \notin \bar{I}} \pi(n_1, \ldots, n_J, \bar{I}) \mu_j(n_j),$$

and the overall throughput of the network is

$$\widetilde{TH}(N; x) = \sum_{i \in \bar{J}} \widetilde{TH}_i(N; x). \tag{5.1}$$

For an in-depth analysis, we first stipulate a convenient form for the throughput. We restrict our presentation to the case of state independent service rates.

Theorem 5.1 *In a Gordon-Newell network with unreliable nodes we have*

$$\widetilde{TH}(N; x) = \frac{\tilde{G}(N-1, \bar{J}; x)}{\tilde{G}(N, \bar{J}; x)}.$$

Proof: We compute the throughput at node 1, the other computations run similar

$$\widetilde{TH}_1(N;x) = \sum_{n\in S(N,\bar{J})} \sum_{\substack{\bar{I}\subset\{2,3\ldots,J\}\\ 1\notin\bar{I}}} \pi(n,\bar{I};x)\mu_1(n_1) = \sum_{n\in S(N,\bar{J})} \sum_{\bar{I}\subset\{2,3\ldots,J\}} \pi(n,\bar{I};x)\mu_1(n_1)$$

$$= \sum_{n\in S(N,\bar{J},)} \tilde{G}(\bar{J},N;x)^{-1} \left(\sum_{\bar{I}\subset\{2,3\ldots,J\}} \frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})} \right)$$

$$\cdot \prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j} \prod_{i=2}^{J}\left(\frac{\eta_i^{n_{ia}+n_{ib}}}{n_{ia}!n_{ib}!\mu_{ia}^{n_{ia}}(x)\mu_{ib}^{n_{ib}}(x)}\right)\mu_1(n_1)$$

$$= \sum_{n\in S(N-1,\bar{J})} \tilde{G}(\bar{J},N;x)^{-1} \left(\sum_{\bar{I}\subset\{2,3\ldots,J\}} \frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})} \right)$$

$$\cdot \prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j} \prod_{i=2}^{J}\left(\frac{\eta_i^{n_{ia}+n_{ib}}}{n_{ia}!n_{ib}!\mu_{ia}^{n_{ia}}(x)\mu_{ib}^{n_{ib}}(x)}\right)\mu_1\frac{\eta_1}{\mu_1}$$

$$= \sum_{n\in S(N-1,\bar{J})} \tilde{G}(\bar{J},N;x)^{-1} \left(\sum_{\bar{I}\subset\{2,3\ldots,J\}} \frac{A(n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})} \right)$$

$$\cdot \prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j} \prod_{i=2}^{J}\left(\frac{h_i(x)^{n_{ia}+n_{ib}}}{n_{ia}!n_{ib}!}\right)\mu_1\frac{\eta_1}{\mu_1}$$

$$= \frac{\tilde{G}(N-1,\bar{J};x)}{\tilde{G}(N,\bar{J};x)}\eta_1$$

□

Our aim is again to maximize the sum of the warehouse throughputs. In general, the problem can not be reduced to the solution of the problem in the network without unreliable nodes from Section 3.

We start with showing that for one customer (=truck) in the system throughput maximization can always be reduced to solving a standard Weber problem. But it is usually not equivalent to problem (2.3) from Section 3.

Example 5.2 *For one customer in the network we have to maximize*

$$\max_{x\in\mathbb{R}^2}\left\{\frac{\tilde{G}(0,\bar{J};x)}{\tilde{G}(1,\bar{J};x)}\eta_1\right\},\ \textit{which is equivalent to minimizing} \min_{x\in\mathbb{R}^2}\left\{\tilde{G}(1,\bar{J};x)\right\},$$

because $\tilde{G}(0,\bar{J};x)$ *and* η_1 *are constants and* $0<\tilde{G}(1,\bar{J};x)<\infty$.

Now

$$
\begin{aligned}
&\tilde{G}(1,\bar{J};x)\\
&=\sum_{n\in S(1,\bar{J})}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\\
&\qquad\qquad\cdot\prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j}\prod_{j=2}^{J}\left(\frac{\eta_j}{\mu_{ja}(x)}\right)^{n_{ja}}\left(\frac{\eta_j}{\mu_{jb}(x)}\right)^{n_{jb}}\\
&=\sum_{\substack{n\in S(1,\bar{J})\\ n_1+\dots+n_J=1}}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j}\\
&\quad+2\sum_{\substack{n\in S(1,\bar{J})\\ n_{2a}+\dots+n_{Jb}=1}}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\prod_{j=2}^{J}(\eta_{ja}d_j(x))^{n_{ja}}\\
&=\sum_{\substack{n\in S(1,\bar{J})\\ n_1+\dots+n_J=1}}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\prod_{j=1}^{J}\left(\frac{\eta_j}{\mu_j}\right)^{n_j}\\
&\quad+2\sum_{j=2}^{J}\left(\sum_{\substack{n\in S(1,\bar{J})\\ n_{ja}=1}}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\right)\eta_{ja}d_j(x).
\end{aligned}
$$

The first summand is a constant, the second is a weighted sum of the distances $d_j(x)$ between the central station 1 and stations $j=2,\dots,J$. The weight of distance $d_j(x)$ is

$$
2\sum_{\substack{n\in S(1,\bar{J})\\ n_{ja}=1}}\left(\sum_{\bar{I}\subseteq\{2,3\dots,J\}}\frac{A(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}{B(\bar{I},n_i,i\in\bigcup_{j\in\bar{I}}\{ja,j,jb\})}\right)\eta_{ja}.
$$

Thus we obtained a Weber problem which is generally not equivalent to problem (2.3), *because it depends on the structure of the demand and no-demand period rates.* □

Detailed computations show that, different from Section 3, the optimal location is not independent of the number of customers in the network.

Even more, in general, our present problem is not a convex optimization problem, as the next example shows.

Example 5.3 *Consider the problem from Section 3 with d the Euclidean distance and $J=3$. Thus, the set of service station consists of $\{1,2a,3a,2,3,2b,3b\}$. Take*

$a_2 := (0,0)$, $a_3 := (0,4)$,
$\mu_1 := 1/4$, $\mu_2 := 1/6$, $\mu_3 := 1/12$,
$r(1,2a) := 2/3$ *and* $r(1,3a) := 1/3$. *So we obtain*
$\mu_{2a}(x) = \mu_{2b}(x) = (x_1^2 + x_2^2)^{-1/2}$ *and* $\mu_{3a}(x) = \mu_{3b}(x) = (x_1^2 + (x_2-4)^2)^{-1/2}$.

The solution of the traffic equation is
$\eta_1 = 1/4$, $\eta_2 = \eta_{2a} = \eta_{2b} = 1/6$ *and* $\eta_3 = \eta_{3a} = \eta_{3b} = 1/12$. *Furthermore*

$$h(x) = \eta_2 d_2(x) + \eta_3 d_3(x) = \frac{1}{6}(x_1^2 + x_2^2)^{1/2} + \frac{1}{12}(x_1^2 + (x_2-4)^2)^{1/2}.$$

$$\text{The optimal solution of the problem is} \quad x^* = \arg\min_{x\in\mathbb{R}^2}\{h(x)\} = (0,0).$$

From Section 2, we know that x^* *maximizes the throughput of the Gordon-Newell network, independent of the number of customers cycling in the network.*

Let us now consider the same network, but with time varying demand at stations $2,3$ *and one respectively two customers in the system. Then the set of possible subsets of frozen stations is* $\{\emptyset, \{2a,2,2b\}, \{3a,3,3b\}, \{2a,2,2b,3a,3,3b\}\}$.
We set $B(\bar I, n_i, i \in \bigcup_{j\in \bar I}\{ja,j,jb\}) := 1$ *for all* $\bar I \subseteq \{2,3\ldots,J\}$ *and all* $n \in S(N,\bar J)$. *To simplify notation we set*

$$\begin{aligned}
A(\bar I_2, n_{2a}, n_2, n_{2b}) &:= A(\{2a,2,2b\}, n_i, i\in\{2,2a,2b\}),\\
A(\bar I_3, n_{3a}, n_3, n_{3b}) &:= A(\{3a,3,3b\}, n_i, i\in\{3a,3,3b\}) \quad \text{and}\\
A(\bar I_{23}, n_{2a}, n_2, n_{2b}, n_{3a}, n_3, n_{3b}) &:= A(\{2a,2,2b,3a,3,3b\}, n_i, i\in\{2a,2,2b,3a,3,3b\}).
\end{aligned}$$

Further we define

$$\begin{aligned}
&A(\bar I_2, n_{2a}, 0, n_{2b}) = A(\bar I_3, n_{3a}, 0, n_{3b}) := 1; && A(\bar I_{23}, n_{2a}, 0, n_{2b}, n_{3a}, 1, n_{3b}) := 1\\
&A(\bar I_2, n_{2a}, 1, n_{2b}) = A(\bar I_3, n_{3a}, 1, n_{3b}) := 1; && A(\bar I_{23}, n_{2a}, 1, n_{2b}, n_{3a}, 1, n_{3b}) := 20\\
&A(\bar I_2, n_{2a}, 2, n_{2b}) = A(\bar I_3, n_{3a}, 2, n_{3b}) := 20; && A(\bar I_{23}, n_{2a}, 2, n_{2b}, n_{3a}, 0, n_{3b}) := 20\\
&A(\bar I_{23}, n_{2a}, 0, n_{2b}, n_{3a}, 0, n_{3b}) := 1; && A(\bar I_{23}, n_{2a}, 0, n_{2b}, n_{3a}, 2, n_{3b}) := 20\\
&A(\bar I_{23}, n_{2a}, 1, n_{2b}, n_{3a}, 0, n_{3b}) := 1. &&
\end{aligned}$$

Then the norming constants with one respectively two customers in the system is:

$$\tilde G(1,\bar 3; x) = 12 + 8h(x) \quad \text{and} \quad \tilde G(2,\bar 3; x) = 79 + 24h(x) + 8h^2(x).$$

Thus with one customer in the system we have to maximize $\widetilde{TH}_1(1;x) = (\tilde G(\bar 3,1,x))^{-1} = (12+8h(x))^{-1}$ *and we get by direct computation* $x^* = (0,0)$ *as solution.*
For two customers $x^* = (0,0)$ *is not the optimal location. In this case have to solve*

$$\max_{x\in\mathbb{R}^2}\left\{\widetilde{TH}(2;x) = \frac{12+8h(x)}{79+24h(x)+8h^2(x)}\right\}.$$

With $h(0,0) = 1/3$ *and* $h(0,-4) = 4/3$ *we get*

$$\widetilde{TH}(2;(0,0)) \approx 0,1669 \quad \text{and} \quad \widetilde{TH}(2;(0,-4)) \approx 0,1919.$$

because there is at least one $x \in \mathbb{R}^2$ *with* $\widetilde{TH}(2,x) > \widetilde{TH}(2,x^*)$, $x^* \in (0,0)$ *is not the optimal solution.*

We can actually show that $\widetilde{TH}$ *is generally not concave. Thus, our problem can not be reduced to a convex optimization problem:*
If $\widetilde{TH}$ *would be concave, then*

$$\widetilde{TH}(2;(1-\lambda)x+\lambda y) \geq (1-\lambda)\widetilde{TH}(2;x)+\lambda\widetilde{TH}(2;y), \quad \forall\, x,y \in \mathbb{R}^2, 0 \leq \lambda \leq 1.$$

With $x=(0,0)$, $y=(0,-4)$ *as well as* $\lambda = 1/2$ *we have*

$$(1-\lambda)x+\lambda y = \frac{1}{2}(0,0)+\frac{1}{2}(0,-4) = (0,-2).$$

Direct computations yield $h(0,-2)=5/6$ *and so* $\widetilde{TH}(2;(0,-2)) \approx 0,1786$. *Otherwise*

$$\begin{aligned}\lambda\widetilde{TH}(2;x)+(1-\lambda)\widetilde{TH}(2;y) &= \frac{1}{2}\widetilde{TH}(2;(0,0))+\frac{1}{2}\widetilde{TH}(2;(0,-4))\\ &\approx \frac{1}{2}0,1669+\frac{1}{2}0,1919 = 0,1794\end{aligned}$$

So we get $TH(2;(1-\lambda)x+\lambda y) \leq (1-\lambda)\widetilde{TH(2;x)}+\lambda\widetilde{TH(2,y)}$.

6 Conclusion and directions of further research

We have developed a throughput optimization method by optimal location of a central server in a star-like network. A prototype example for such models is our introductory example, which is a location problem in the continuous plane.

Starting from an exponential version of the problem which admits a simple proof of the solution we have shown that much more realistic models are covered by the result.

Probably, the most important result of our investigation is that we can to a certain extent decouple the decision about location of the central service facility (production center) and the building of the exterior stations.

Clearly, many research problems are not yet tackled in the more general area of location of additional nodes in networks of queues. These can be easily identified by investigating the more involved location problems, like p-median or p-center problems, in the continuous as well as in the discrete setting in networks of queues.

Parts of our ongoing research related to the present paper are
(i) location theory in the discrete queueing network setting with prescribed network graphs, and (i) p-center location problems in the setting of this paper.

A Appendix: Proofs

Recalling the denomination of the node set $\bar{J} = \{1, 2a, 3a, \ldots, Ja, 2, 3, \ldots, J, 2b, 3b, \ldots, Jb\}$, for preparing the proof of Theorem 2.1 we specify first the probability solution of the traffic equation (3.1) by direct solution as

$$\eta_1 = \frac{1}{4} \quad \text{and} \quad \eta_j = \eta_{ja} = \eta_{jb} = \frac{1}{4} r(1, ja) = \frac{1}{4}\rho(j), \quad j = 2, \ldots, J, \tag{A.1}$$

and notice that the stationary distribution of the network is $\pi(n_j, j \in \bar{J}; x) =$

$$G^{-1}(N, \bar{J}; x) \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \prod_{i=2}^{J} \left(\prod_{l=1}^{n_{ia}} \left(\frac{\eta_{ia}}{l\mu_{ia}(x)} \right) \prod_{m=1}^{n_{ib}} \left(\frac{\eta_{ib}}{m\mu_{ib}(x)} \right) \right)$$

$$= G^{-1}(N, \bar{J}; x) \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \prod_{i=2}^{J} \frac{1}{n_{ia}! n_{ib}!} (\eta_i d_i(x))^{n_{ia}+n_{ib}}.$$

where we utilized $\eta_{ja} = \eta_{jb} = \eta_j$ and $\mu_{ja}(x) = \mu_{jb}(x) = d_j^{-1}(x)$ for $j = 2, \ldots, J$. Because we express throughput by norming constants the following representation will be of value.

Lemma A.1 *The norming constant of the system is*

$$G(N, \bar{J}; x) = \sum_{n=0}^{N} \left[\left(\sum_{n_1+\ldots+n_J=N-n} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \right) \frac{2^n}{n!} \left(\sum_{j=2}^{J} \eta_j d_j(x) \right)^n \right].$$

With

$$C_n(N, \bar{J}) := \left(\sum_{n_1+\ldots+n_J=N-n} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \right) \frac{2^n}{n!}$$

and

$$h_j(x) := \eta_j d_j(x), \qquad h(x) := \sum_{j=2}^{J} h_j(x) = \sum_{j=2}^{J} \eta_j d_j(x)$$

we can write

$$G(N, \bar{J}; x) = \sum_{n=0}^{N} C_n(N, \bar{J}) h(x)^n. \tag{A.2}$$

Proof: From the definition we have $G(N, \bar{J}; x) =$

$$= \sum_{n_1+\ldots+n_{Jb}=N} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \prod_{i=2}^{J} \frac{(\eta_i d_i(x))^{n_{ia}+n_{ib}}}{n_{ia}! n_{ib}!}$$

$$= \sum_{n=0}^{N} \left[\left(\sum_{n_1+\ldots+n_J=N-n} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)} \right) \right) \left(\sum_{\substack{n_{2a}+\cdots+n_{Ja}+ \\ n_{2b}+\cdots+n_{Jb}=n}} \prod_{j=2}^{J} \frac{h_j(x)^{n_{ja}+n_{jb}}}{n_{ja}! n_{jb}!} \right) \right].$$

The statement follows from

$$\sum_{\substack{n_{2a}+\ldots+n_{Ja}+\\ n_{2b}+\ldots+n_{Jb}=n}} \prod_{j=2}^{J} \frac{h_j(x)^{n_{ja}+n_{jb}}}{n_{ja}!n_{jb}!} = \sum_{\substack{n_{2a}+\ldots+n_{Ja}+\\ n_{2b}+\ldots+n_{Jb}=n}} \prod_{j=2}^{J} h_j(x)^{n_{ja}} \prod_{i=2}^{J} h_i(x)^{n_{ib}} \frac{1}{\prod_{i=2}^{J} n_{ia} \prod_{i=2}^{J} n_{ib}}$$

$$= \sum_{\substack{n_{2a}+\ldots+n_{Ja}+\\ n_{2b}+\ldots+n_{Jb}=n}} \underbrace{\prod_{j=2}^{J} \left(\frac{h_j(x)}{2h(x)}\right)^{n_{ja}} \prod_{i=2}^{J} \left(\frac{h_i(x)}{2h(x)}\right)^{n_{ib}} \frac{n!}{\prod_{i=2}^{J} n_{ia} \prod_{i=2}^{J} n_{ib}}}_{\text{density function of a multinomial distribution}} \cdot \frac{(2h(x))^n}{n!} = \frac{2^n}{n!} h(x)^n.$$

□

Proof: (**of Theorem 3.8**) The representation of the norming constant is the first statement of Lemma A.1, the throughput (3.6) is the standard result in (3.3), and (3.7) follows from (3.2) and (A.1). □

Before starting with the proof of Theorem 2.1 we highlight the value of the representation (A.2) by considering the case $N = 1$, where we get from Lemma A.1

$$\begin{aligned} x^* = \arg\max_{x\in\mathbb{R}^2} \{TH(1;x)\} \Leftrightarrow x^* &= \arg\max_{x\in\mathbb{R}^2} \left\{\frac{G(0,\bar{J};x)}{G(1,\bar{J};x)}\right\} \\ \Leftrightarrow x^* &= \arg\max_{x\in\mathbb{R}^2} \left\{\frac{1}{C_0(1,\bar{J}) + C_1(1,\bar{J})h(x)}\right\} \\ \Leftrightarrow x^* &= \arg\max_{x\in\mathbb{R}^2} \{(C_0(1,\bar{J}) + C_1(1,\bar{J})h(x))^{-1}\} \\ \Leftrightarrow x^* &= \arg\min_{x\in\mathbb{R}^2} \{C_0(1,\bar{J}) + C_1(1,\bar{J})h(x)\}. \end{aligned}$$

$C_0(1,\bar{J})$ and $C_1(1,\bar{J})$ are constants, which leads to our minimization problem (2.3)

$$\begin{aligned} x^* = \arg\max_{x\in\mathbb{R}^2} \{TH(1;x)\} \quad &\Leftrightarrow \quad x^* = \arg\min_{x\in\mathbb{R}^2} \{h(x)\} \\ \text{with } \ h(x) \quad &= \quad \sum_{j=2}^{J} \eta_j d_j(x) = \frac{1}{4}\sum_{j=2}^{J} r(1,ja) d_j(x) = \frac{1}{4}\sum_{j=2}^{J} \rho(j) d_j(x). \end{aligned}$$

This is the classical Weber problem with $J-1$ demand points and weight $r(1,ja)$ at demand point a_j. Hence this problem can be solved for instance by using the Weiszfeld-procedure.

The following lemma will be used.

Lemma A.2 *For all $k = 0,1,\ldots,N-2$, $n = 0,1,\ldots,N-k-2$ holds*

$$C_{n+k+1}(N,\bar{J})C_k(N-1,\bar{J}) \geq C_{n+k+1}(N-1,\bar{J})C_k(N,\bar{J}).$$

Proof: By definition

$$C_k(N,\bar{J}) = \left(\sum_{n_1+\ldots+n_J=N-k} \prod_{j=1}^{J} \left(\prod_{k=1}^{n_j} \frac{\eta_j}{\mu_j(k)}\right)^{n_j}\right) \frac{2^k}{k!},$$

hence, with Theorem 3.2, $C_k(N, \bar{J})$ is the norming constant $G(N-k, J)$ of a standard Gordon-Newell network with $N-k$ customers, J service stations and not normalized solution $(\eta_1, \ldots, \eta_J)$ of the associated traffic equation, multiplied with $2^k/k!$.
With $\sum_{j=1}^{J} \eta_j = 1/2$ the throughput of this standard Gordon-Newell network with N customers IS $TH(N) = (1/2)G(N-1, J)/G(N, J)$, see (3.4). Therefore we get

$$\begin{aligned} & C_{n+k+1}(N, \bar{J})C_k(N-1, \bar{J}) \geq C_{n+k+1}(N-1, \bar{J})C_k(N, \bar{J}) \\ \Leftrightarrow\ & G(N-n-k-1, J)G(N-k-1, J) \geq G(N-n-k-2, J)G(N-k, J) \\ \Leftrightarrow\ & \frac{G(N-k-1, J)}{G(N-k, J)} \geq \frac{G(N-n-k-2, J)}{G(N-n-k-1, J)} \\ \Leftrightarrow\ & TH(N-k) \geq TH(N-n-k-1). \end{aligned}$$

We know from Theorem 3.6 [WA89] that the throughput of a Gordon-Newell-network with service rates which are nondecreasing in the number of customers at the node is a nondecreasing function in the network's population size. Thus the lemma is proved. □

Proof: (of Theorem 2.1) For all $x \in \mathbb{R}^2$ with $h(x) > h(x^*)$ and $N \in \mathbb{N}$ we have to show

$$TH(N; x^*) = \frac{G(N-1, \bar{J}; x^*)}{G(N, \bar{J}; x^*)} > \frac{G(N-1, \bar{J}; x)}{G(N, \bar{J}; x)} = TH(N; x)$$

By Lemma A.1 this is equivalent to

$$\begin{aligned} & G(N-1, \bar{J}; x^*)G(N, \bar{J}; x) - G(N-1, \bar{J}; x)G(N, \bar{J}; x^*) > 0 \\ \Leftrightarrow\ & \left(\sum_{n=0}^{N-1} C_n(\mathbb{N}-1, \bar{J})h(x^*)^n)\right)\left(\sum_{k=0}^{N} C_k(N, \bar{J})h(x)^k\right) \\ -\ & \left(\sum_{n=0}^{N-1} C_n(N-1, \bar{J})h(x)^n\right)\left(\sum_{k=0}^{N} C_k(N, \bar{J})h(x^*)^k)\right) > 0 \\ \Leftrightarrow\ & \sum_{k=0}^{N}\sum_{n=0}^{N-1} C_k(N, \bar{J})C_n(N-1, \bar{J})\left(h(x)^k h(x^*)^n - h(x)^n h(x^*)^k\right) > 0. \end{aligned}$$

We consider the summand for $k = N$

$$\begin{aligned} & \sum_{n=0}^{N-1} C_N(N, \bar{J})C_n(N-1, \bar{J})\left(h(x)^N h(x^*)^n - h(x)^n h(x^*)^N\right) \\ =\ & \sum_{n=0}^{N-1} C_N(N, \bar{J})C_n(N-1, \bar{J})h(x)^n h(x^*)^n\left(h(x)^{N-n} - h(x^*)^{N-n}\right). \end{aligned}$$

Because of $h(x) > h(x^*)$ we have $h(x)^{N-n} - h(x^*)^{N-n} > 0$. So, the whole summand is strictly positive and the problem is reduced to prove

$$\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} C_k(N, \bar{J})C_n(N-1, \bar{J})(h(x)^k h(x^*)^n - h(x)^n h(x^*)^k) > 0.$$

For $k = n$ we get $h(x)^k h(x^*)^n - h(x)^n h(x^*)^k = 0$. So, the problem is reduced to

$$\sum_{k=0}^{N-1} \sum_{\substack{n=0 \\ n \neq k}}^{N-1} C_k(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^k h(x^*)^n - h(x)^n h(x^*)^k) > 0$$

$$\Leftrightarrow \sum_{k=0}^{N-1} \sum_{n=k+1}^{N-1} C_k(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^k h(x^*)^n - h(x)^n h(x^*)^k) +$$
$$\sum_{k=0}^{N-1} \sum_{n=0}^{k-1} C_k(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^k h(x^*)^n - h(x)^n h(x^*)^k) > 0.$$

Now we consider the case $k = N - 1$ in the first summand. Then the second sum is empty and therefore zero. The same argument holds in the second summand for $k = 0$ and we have reduce the problem further to

$$\sum_{k=0}^{N-2} \sum_{n=k+1}^{N-1} C_k(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^k h(x^*)^n - h(x)^n h(x^*)^k)$$
$$+ \sum_{k=1}^{N-1} \sum_{n=0}^{k-1} C_k(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^k h(x^*)^n - h(x)^n h(x^*)^k) > 0.$$

by index-shift in both summands we get

$$\sum_{k=0}^{N-2} \sum_{n=0}^{N-k-2} C_k(N, \bar{J}) C_{n+k+1}(N-1, \bar{J}) (h(x)^k h(x^*)^{n+k+1} - h(x)^{n+k+1} h(x^*)^k)$$
$$+ \sum_{k=0}^{N-2} \sum_{n=0}^{k} C_{k+1}(N, \bar{J}) C_n(N-1, \bar{J}) (h(x)^{k+1} h(x^*)^n - h(x)^n h(x^*)^{k+1}) > 0.$$

In the second summand we apply twice the following summation formula:
for $a_k, b_k \in \mathbb{R}$, $k = 1, \ldots, N$,

$$\sum_{k=0}^{N} \sum_{n=0}^{k} a_{k+1} b_n = \sum_{k=0}^{N} \sum_{n=0}^{N-k} a_{n+k+1} b_k,$$

and obtain

$$\sum_{k=0}^{N-2} \sum_{n=0}^{N-k-2} C_k(N, \bar{J}) C_{n+k+1}(N-1, \bar{J}) (h(x)^k h(x^*)^{n+k+1} - h(x)^{n+k+1} h(x^*)^k)$$
$$+ \sum_{k=0}^{N-2} \sum_{n=0}^{N-k-2} C_{n+k+1}(N, \bar{J}) C_k(N-1, \bar{J}) (h(x)^{n+k+1} h(x^*)^k - h(x)^k h(x^*)^{n+k+1}) > 0.$$

Because of $h(x) > h(x^*)$ we have $h(x)^{n+k+1} h(x^*)^k - h(x)^k h(x^*)^{n+k+1} > 0$ and from Lemma A.2 we have $C_{n+k+1}(N, \bar{J}) C_k(N-1, \bar{J}) \geq C_{n+k+1}(N-1, \bar{J}) C_k(N, \bar{J})$. So we have

$$\begin{aligned}
& \sum_{k=0}^{N-2}\sum_{n=0}^{N-k-2} C_k(N,\bar{J})C_{n+k+1}(N-1,\bar{J})(h(x)^k h(x^*)^{n+k+1} - h(x)^{n+k+1}h(x^*)^k) \\
+ & \sum_{k=0}^{N-2}\sum_{n=0}^{N-k-2} C_{n+k+1}(N,\bar{J})C_k(N-1,\bar{J})(h(x)^{n+k+1} h(x^*)^{k} - h(x)^{k}h(x^*)^{n+k+1}) \\
\geq & \sum_{k=0}^{N-2}\sum_{n=0}^{N-k-2} C_k(N,\bar{J})C_{n+k+1}(N-1,\bar{J})(h(x)^k h(x^*)^{n+k+1} - h(x)^{n+k+1}h(x^*)^k) \\
+ & \sum_{k=0}^{N-2}\sum_{n=0}^{N-k-2} C_k(N,\bar{J})C_{n+k+1}(N-1,\bar{J})(h(x)^{n+k+1} h(x^*)^{k} - h(x)^{k}h(x^*)^{n+k+1}) \\
= & \ 0,
\end{aligned}$$

and so the theorem is proved. □

Literatur

[BLC85] Berman, O., Larson, R.C., Chiu, S.S. (1985) „Optimal Server Location on a Network Operating as an M/G/1 Queue“, Operations Research 33 , 746-771.

[BLP87] Berman, O., Larson, R.C., Parkan, C. (1987) „The Stochastic Queue p-Median Problem“, Transportation Science 21, 207-216.

[CY01] Chen, H.,Yao, D.D. (2001) *Fundamentals of Queueing Networks, Performance, Asymptotics and Optimization,* Stochastic Modelling and Applied Probability 46, Springer, New York.

[DA01] Daduna, H. (2001) „Stochastic Networks with Product Form Equilibrium“, in: Shanbhag, D. N., Rao, C. R. (eds.) *Handbook of Statistics, Vol. 19* Elsevier, Amsterdam, 309 - 364.

[DH04] Drezner, Z., Hamacher,H. W. (eds.) (2004) *Facility Location, Applications and Theory,* Springer, Berlin, 1. ed., 2. Printing.

[DSS90] Drezner, Z., Schaible, S., Simchi-Levi, D. (1990) „Queueing-Location Problems on the Plane“, *Naval Research Logistics 37*, 929-935.

[DKLSW04] Drezner, Z., Klamroth, K., Schoebel, A., WesolowskyG. O. (2004) „The Weber Problem“, in: Drezner,Z., Hamacher,H. (eds.) *Facility Location, Applications and Theory,* Springer, Berlin, 1. ed., 2. Printing.

[GN67] Gordon, W.J., Newell,G.F. (1967) „Closed queueing networks with exponential servers“, *Operations Research 15*, 254-265.

[JA63] Jackson, J.R., (1963) „Jobshop-like queueing systems“, *Management Science 10*, 131-142.

[LA74] Larson, R.C. (1974) „A Hypercube Qeuing Model for Facility Location and Redistricting in Urban Emergency Services“, *Computers and Operations Research 1*, 67-95.

[MF90] Mirchandani, P.B., Francis, R.L. (eds.) (1990) *Discret Location Theory,* Wiley, New York.

[SA06] Sauer, C., (2006) *Stochastic Product Form Networks with Unreliable Nodes: Analysis of Performance and Availability*, Dissertation, Universität Hamburg.

[SD03] Sauer, C., Daduna, H. (2003) „Availability formulas and performance measures for separable degradable networks“, *Economic Quality Control 18*, 165 - 194.

[SJD99] Scott, C., Jefferson, T., Drezner, Z. (1999) „Various Objectives for the Queueing-Location Problem on the Plane“, *Asia-Pacific Journal of Operational Research 16*, 203-214.

[SS94] Shaked, M., Shanthikumar, J.G. (1994) *Stochastic Orders and Their Applications* Academic Press, Boston.

[WA89] Wal, J. van der (1989) „Monotonicity of the Throughput of a Closed Exponential Queueing Network in the Number of Jobs“, *OR-Spektrum 11*, 97-100.

[WE09] Weber, A. (1909) *Über den Standort der Industrien,* Tübingen.

[WE37] Weiszfeld, E. (1937) „Sur le point par lequel la somme des distances de n points donnés est minimum“, *Tohoku Mathematics Journal 43*, 355-386.